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ON A PROBLEM OF GEÖCZE.*

By TIBOR RADÓ.

1. Introduction.

1.1. The Lebesgue area $A(S)$ of a continuous surface S is defined by the formula

$$A(S) = gr.\,l.\,b.\,\underline{\lim}\, E(\Pi_n),$$

where the greatest lower bound is taken with respect to all sequences of polyhedra Π_n converging to S in the Fréchet sense, and $E(\Pi_n)$ is the area of Π_n in the elementary sense.[1]

1.2. Let us denote by $A^*(S)$ the quantity obtained by requiring, in the preceding definition, that the polyhedra Π_n be *inscribed* in S. By restricting the polyhedra, we certainly do not decrease the greatest lower bound. Hence $A^*(S) \geqq A(S)$. Obviously then, we have surely $A^*(S) = A(S)$ if $A(S) = \infty$.

1.3. It may be argued that it would be more natural to consider $A^*(S)$, defined in terms of *inscribed* polyhedra, as the area of S. Still, $A(S)$ is generally accepted as the area, probably because very little is known about $A^*(S)$. The problem of determining whether $A^*(S) = A(S)$ may properly be called *the problem of Geöcze*, because Geöcze was the first to study this question in a substantial manner. Several of his many profound papers on the area of surfaces are devoted to this subject.[2] The purpose of the present paper is to improve the result obtained by Geöcze concerning this problem.

1.4. We shall be concerned with a surface S of the special form

$$S: \quad z = f(x, y), \quad (x, y) \in I_0, \tag{1}$$

where I_0 is the unit square

$$I_0: \quad 0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1,$$

and $f(x, y)$ is *continuous* in I_0. For $0 \leqq x \leqq 1$, let $L(x)$ denote the length of the arc $x = \text{const.}$, $0 \leqq y \leqq 1$, $z = f(x, y)$. Similarly, let us denote, for

* Received March 9, 1942; Presented to the American Mathematical Society at the meeting in Chicago, April 1942.

[1] For general information concerning the Lebesgue area of a surface, see for example Radó [1], Chapter I.

[2] See Geöcze [1], [2], [3].

$0 \leqq y \leqq 1$, by $M(y)$ the length of the arc $y = \text{const.}$, $0 \leqq x \leqq 1$, $z = f(x, y)$. Geöcze proved the equality $A^*(S) = A(S)$ under the following assumptions.[3]

(i) The functions $L(x)$, $M(y)$ are bounded.

(ii) $f(x, y)$ satisfies a uniform Lipschitz condition with respect to one of the variables, say x. That is, there exists a finite constant L_0 such that

$$|f(x_2, y) - f(x_1, y)| \leqq L_0 |x_2 - x_1|$$

for all $0 \leqq x_1 \leqq 1$, $0 \leqq x_2 \leqq 1$, $0 \leqq y \leqq 1$.

1.5. We shall extend this result by proving the following theorem.

THEOREM. *Suppose that the function $f(x, y)$, defining the continuous surface (1), is absolutely continuous with respect to x for a. e. y in $0 \leqq y \leqq 1$ (or, alternatively, that $f(x, y)$ is absolutely continuous with respect to y for a. e. x in $0 \leqq x \leqq 1$). Then $A^*(S) = A(S)$.*

1.6. Let us recall that $A^*(S) = A(S)$ if $A(S) = \infty$ (see **1.2**). In proving the preceding theorem we can therefore assume that $A(S) < \infty$. But then, by a result of Geöcze,[4] the functions $L(x)$, $M(y)$ of **1.4.** are summable. Hence, in proving our theorem, we shall really use the following assumptions.

(i)* The functions $L(x)$, $M(y)$ are summable.

(ii)* For a. e. y in $0 \leqq y \leqq 1$, the function $f(x, y)$ is absolutely continuous with respect to x.

Stated in this manner, our assumptions may be more readily compared with those of Geöcze (see **1.4**).

1.7. Geöcze stated that he found a proof of $A^*(S) = A(S)$, for surfaces of the form (1), without any restriction upon $f(x, y)$ beyond *mere continuity*.[5] He also stated that his proof depended upon a thorough study of Riemann-type approximations to Lebesgue integrals and also upon a topological study of the sections $z = \text{const.}$ of the surface $z = f(x, y)$. While he did publish extensive and profound investigations in both of these subjects,[6] the present writer was unable to find in his published work either a general proof of the theorem $A^*(S) = A(S)$ or sufficient information to construct such a proof. The proof presented in this paper for the partial result stated in **1.5** makes essential use of the work of Geöcze on Riemann-type approximations, but the author was unable to utilize the topological results of Geöcze on the sections

[3] See Geöcze [1].
[4] See Geöcze [4].
[5] See Geöcze [2].
[6] See Geöcze [2], [3].

$z =$ const. of the surface which he stated were essential for his proof. For this reason, it is unlikely that our proof is similar to the one Geöcze had in mind for the general theorem.

1.8. The method followed in this paper, in proving the theorem of **1.5**, is essentially a refinement of the method used by Geöcze [7] to prove the result stated in **1.4**. Various changes were necessary on account of the greater generality of our assumptions, and substantial simplifications were achieved by properly modifying some of the auxiliary quantities introduced by Geöcze (see especially **3.7**). However, a definite effort was made to disturb as little as possible the original line of thought of Geöcze.

1.9. The literature on the area of surfaces contains many beautiful results concerning approximations by inscribed polyhedra. As far as the author is aware, no result of a generality comparable to that of our theorem in **1.5.** has been previously obtained. For this reason, we restrict ourselves to a reference to the work of Kempisty [1, 2], because he proved several results of great interest beyond the mere equality $A^*(S) = A(S)$. On the other hand, his assumptions are far more restrictive than ours (cf. Kempisty [1]).

2. On Riemann-type approximations.

2.1. In studying the continuous surface S given by (1), we can and shall assume that $A(S) < \infty$ (cf. **1.6**). As a consequence, condition (i)* of **1.6** is satisfied. Furthermore, condition (ii)* in **1.6** will be assumed throughout the paper.

2.2. X will designate a subdivision of the interval $0 \leqq x \leqq 1$ of the form

$$X: \quad 0 = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = 1.$$

It will be convenient to describe the generic interval of X by writing $x' \leqq x \leqq x''$. Henceforth, x' will always refer to the left-hand end-point and x'' to the right-hand end-point of a generic interval of X. The following symbols will be used. $\| X \|$ will designate the norm of X, that is the maximum length of the sub-intervals of X, $q(X)$ the number of intervals of X, and $p(X)$ the parameter of regularity of X, that is the ratio of the longest to the shortest interval of X. Thus always $p(X) \geqq 1$. We shall also use subdivisions Y of the interval $0 \leqq y \leqq 1$ and the corresponding symbols y', y'', $\| Y \|$, $q(Y)$, $p(Y)$. A pair of subdivisions X, Y determine, in an obvious manner, a subdivision into rectangles of the unit square I_0: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. This subdivision will be denoted by XY.

---

[7] See Geöcze [1].

2.3. Let $\psi(\Delta)$ be an interval-function defined for all closed sub-intervals Δ in $0 \leqq x \leqq 1$. Then X_ψ will denote the sum of the values of $\psi(\Delta)$ for the sub-intervals of X. The symbol Y_ψ will be used in a similar sense. If $\psi(r)$ is a rectangle-function defined for all closed rectangles $r: x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ in the unit square I_0, then XY_ψ will denote the sum of the values of $\psi(r)$ for the rectangles of the subdivision XY.

2.4. Let us consider now the function $L(x)$ of **1.4**. Since $L(x)$ is summable in $0 \leqq x \leqq 1$, it is finite a. e. in this interval. We agree once and for all to use only subdivisions X such that $L(x)$ is finite at every interior point of division of X. Of course, $L(0)$ and $L(1)$ may be infinite. For this reason, it may happen that in certain summations X_ψ the terms depending upon $x' = 0$ or $x'' = 1$ are not finite. In such a case, we shall write X^*_ψ, instead of X_ψ, to indicate that such exceptional terms are to be discarded, and hence their influence should be studied separately. Similar conventions will apply to the symbols Y, Y^* in relation to the function $M(y)$ of **1.4**.

An important example is furnished by the summations

$$X^*L(x')(x''-x'), \qquad X^*L(x'')(x''-x'),$$
$$Y^*M(y')(y''-y'), \qquad Y^*M(y'')(y''-y').$$

It will be a matter of great importance for us to know that by proper choice of X and Y these summations may be brought as close as we wish to

$$\int_0^1 L(x)\,dx, \qquad \int_0^1 M(y)\,dy$$

respectively, and that furthermore X, Y may be made to satisfy a variety of further conditions necessary in the sequel. The required facts will be summarized presently.[8]

2.5. Let there be given two constants ϵ, p such that $0 < \epsilon < 1$, $p > 1$. Then we have an $\eta > 0$, $0 < \eta < 1$, such that the following statements hold. Given any measurable subset e_x of $0 \leqq x \leqq 1$ such that $|e_x| < \eta$, and any measurable subset e_y of $0 \leqq y \leqq 1$ such that $|e_y| < \eta$, there exists a sequence of subdivisions $X_\nu Y_\nu$ of the unit square I_0, depending upon ϵ, p, e_x, e_y and the functions $L(x)$, $M(y)$, with the following properties.

(a) $\|X_\nu\| \to 0, \qquad \|Y_\nu\| \to 0.$

(b) $q(X_\nu) = q(Y_\nu), \qquad (\nu = 1, 2, \cdots).$

(c) $p(X_\nu) < p, \qquad p(Y_\nu) < p, \qquad (\nu = 1, 2, \cdots).$

---

[8] Except for slight changes in wording, these facts are proved in Geöcze [3]. Subsequently, these results of Geöcze were extended in Hahn [1].

(d) For each ν, $X_{\nu+1}$ is a refinement of X_ν and $Y_{\nu+1}$ is a refinement of Y_ν.

(e) $L(x)$ is finite at every interior point of division of X_ν, and $M(y)$ is finite at every interior point of division of Y_ν.

(f) $$\underset{x'\epsilon c_x}{X^*_\nu} L(x')(x''-x') < \epsilon, \quad \underset{x''\epsilon c_x}{X^*_\nu} L(x'')(x''-x') < \epsilon,$$
$$\underset{y'\epsilon c_y}{Y^*_\nu} M(y')(y''-y') < \epsilon, \quad \underset{y''\epsilon c_y}{Y^*_\nu} M(y'')(y''-y') < \epsilon.$$

(g) $$|X^*_\nu L(x')(x''-x') - \int_0^1 L(x)dx| < \epsilon,$$
$$|X^*_\nu L(x'')(x''-x') - \int_0^t L(x)dx| < \epsilon,$$
$$|Y^*_\nu M(y')(y''-y') - \int_0^1 M(y)dy| < \epsilon,$$
$$|Y^*_\nu M(y'')(y''-y') - \int_0^1 M(y)dy| < \epsilon.$$

2.6. In the sequel, ϵ and p will be arbitrarily assigned and then kept fixed. Next, a corresponding η, in the sense of **2.5**, will be taken and then kept fixed. Then the sets e_x and e_y will be selected in the following manner. $L(x)$ being summable, we can select e_x to satisfy the following conditions. (1) $|e_x| < \eta$. (2) The complement E_x of e_x, with respect to $0 \leqq x \leqq 1$, is a closed set. (3) $L(x)$ is finite and continuous on E_x.[9]

2.7. The selection of e_y is more complicated, since we shall bring into play condition (ii)* of **1.6**. Since $M(y)$ is summable, we can first select a set e'_y in $0 \leqq y \leqq 1$ to satisfy the following conditions. (1) $|e'_y| < \eta/3$. (2) The complement E'_y of e'_y with respect to $0 \leqq y \leqq 1$ is a closed set. (3) $M(y)$ is finite and continuous on E'_y. Next, since $f(x,y)$ is absolutely continuous with respect to x for a. e. y in $0 \leqq y \leqq 1$, we can select a subset e''_y of E'_y with the following properties. (1) $|e''_y| < \eta/3$. (2) The set $E''_y = E'_y - e''_y$ is closed. (3) For every $y \epsilon E''_y$, $f(x,y)$ is absolutely continuous with respect to x. Take now a $y \epsilon E''_y$. Since then $f(x,y)$ is absolutely continuous with respect to x, for every interval $x' \leqq x \leqq x''$ the length s of the arc $y = y$, $x' \leqq x \leqq x''$, $z = f(x,y)$ is given by

$$s = \int_{x'}^{x''} [1 + (\partial f/\partial x)^2]^{1/2} dx. \tag{2}$$

Consider then any finite system of such arcs, with lengths $s_1, s_2, \cdots$, corresponding to the given $y \epsilon E''_y$ and to a system of non-overlapping intervals

[9] See Saks [1], p. 72.

$x_1' \leqq x \leqq x_1''$, $x_2' \leqq x \leqq x_2'', \cdots$ of total length $\leqq 1/k$, where k is a positive integer. Let $\omega_k(y)$ designate the least upper bound of $s_1 + s_2 + \cdots$ for all such systems, with y and k fixed. By the absolute continuity of the indefinite Lebesgue integral, it follows from (2) that for fixed $y \in E_y''$

$$\omega_k(y) \to 0 \quad \text{for} \quad k \to \infty.$$

Further, for fixed k, $\omega_k(y)$ is clearly a lower semi-continuous, and hence measurable, function of $y \in E_y''$. Hence we have a subset e_y''' of E_y'' with the following properties. (1) $|e_y'''| < \eta/3$. (2) The set $E_y''' = E_y'' - e_y'''$ is closed. (3) $\omega_k(y)$ is continuous on E_y''', $k = 1, 2, \cdots$. Thus we have, on the closed set E_y''', a decreasing sequence of continuous functions $\omega_k(y)$ converging to zero. By the theorem of Dini, the convergence of such a sequence is *uniform*.[10] Hence if we define, for $\psi > 0$,

$$\omega(\psi) = \text{l. u. b.}_{\substack{y \in E_y''' \\ 1/k \leqq \psi}} \omega_k(y)$$

then $\omega(\psi) \to 0$ for $\psi \to 0$.

2.8. We put now $e_y = e_y' + e_y'' + e_y'''$. Then the complement of e_y, with respect to $0 \leqq y \leqq 1$, is the set E_y''' which we shall denote by E_y, and the following statements hold.

(1) $|e_y| < \eta$.

(2) E_y, the complement of e_y with respect to $0 \leqq y \leqq 1$, is closed.

(3) $M(y)$ is finite and continuous on E_y.

(4) We have a function $\omega(\psi)$, defined for $\psi > 0$, such that $\omega(\psi) \to 0$ for $\psi \to 0$, and such that the following holds. If $y_0 \in E_y$, and if $x_1' \leqq x \leqq x_1''$, $x_2' \leqq x \leqq x_2'', \cdots$ are non-overlapping intervals in $0 \leqq x \leqq 1$ with total length $\leqq \psi$, then the sum of the lengths of the arcs

$$\begin{aligned} &y = y_0, x_1' \leqq x \leqq x_1'', \qquad z = f(x, y_0), \\ &y = y_0, x_2' \leqq x \leqq x_2'', \qquad z = f(x, y_0), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

is $\leqq \omega(\psi)$.

2.9. For given $0 < \epsilon < 1$ and $p > 1$, and a corresponding $\eta > 0$, we have now selected the sets e_x, e_y, whose complements, with respect to $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$ respectively, we denoted by E_x, E_y. Thereupon we select a sequence $X_\nu Y_\nu$ as described in **2.5**. Until further notice, ϵ, p, e_x, e_y, E_x, E_y, $\omega(\psi)$, $X_\nu Y_\nu$ will be kept fixed. In the sequel, we shall use the Landau symbol O in writing relations of the form $A = O(B)$, with the following meaning:

[10] This is a well-known theorem of Dini. See any standard text on Real Variables.

there exists some finite constant $\Gamma > 0$, depending only upon the given constant $p > 1$ and the given function $z = f(x, y)$, but *independent* of ϵ, η, e_x, e_y, E_x, E_y, $\omega(\psi)$, $X_\nu Y_\nu$, such that $A \leqq \Gamma B$.

2.10. For the sequence $X_\nu Y_\nu$ we have (see **2.5**, (b)) $q(X_\nu) = q(Y_\nu)$. Let us designate this common value by q_ν. From the conditions (a), (b), (c) in **2.5** we infer easily the following relations for the generic intervals $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ of X_ν, Y_ν respectively.

$$x'' - x' = O(1/q_\nu), \quad y'' - y' = O(1/q_\nu), \quad (x'' - x')(y'' - y') = O(1/q_\nu^2),$$

$$\frac{1}{x'' - x'} = O(q_\nu), \quad \frac{1}{y'' - y'} = O(q_\nu), \quad \frac{1}{(x'' - x')(y'' - y')} = O(q_\nu^2),$$

$$x'' - x' = O(y'' - y'), \quad y'' - y' = O(x'' - x').$$

2.11. The sequence $X_\nu Y_\nu$ gives rise to two sequences of functions $l_\nu(x)$, $m_\nu(y)$ defined as follows:

$$l_\nu(x) = Y_\nu[(y'' - y')^2 + (f(x, y'') - f(x, y'))^2]^{1/2}, \qquad 0 \leqq x \leqq 1,$$

$$m_\nu(y) = X_\nu[(x'' - x')^2 + (f(x'', y) - f(x', y))^2]^{1/2}, \qquad 0 \leqq y \leqq 1.$$

Thus $l_\nu(x)$, $m_\nu(y)$ represent the lengths of polygons inscribed in the sections $x = \text{const.}$, $y = \text{const.}$ respectively of the surface (1). By condition (d) in **2.5** we have therefore

$$l_1(x) \leqq l_2(x) \leqq \cdots, \tag{3}$$

and, by the geometrical meaning of $l_\nu(x)$,

$$l_\nu(x) \to L(x) \quad \text{for} \quad \nu \to \infty. \tag{4}$$

Since $l_\nu(x)$ is obviously a continuous function of x, and since $L(x)$ is continuous on the closed set E_x, (3) and (4) imply, by the theorem of Dini, that $l_\nu(x)$ converges to $L(x)$ uniformly on E_x.[11] Similarly, it follows that $m_\nu(y)$ converges to $M(y)$ uniformly on E_y. Therefore, if we define

$$\delta_\nu = \max_{x \in E_x} (L(x) - l_\nu(x)) + \max_{y \in E_y} (M(y) - m_\nu(y)),$$

then

$$\delta_\nu \to 0 \quad \text{for} \quad \nu \to \infty \tag{5}$$

and

$$\begin{aligned} L(x) - l_\nu(x) &\leqq \delta_\nu, \qquad x \in E_x, \\ M(y) - m_\nu(y) &\leqq \delta_\nu, \qquad y \in E_y. \end{aligned} \tag{6}$$

2.12. Let d_ν designate the greatest one of the oscillations of $f(x, y)$ in

[11] See [10].

the various rectangles of $X_\nu Y_\nu$. From (a) in **2.5** it follows that $d_\nu \to 0$ for $\nu \to \infty$.

2.13. By **2.6**, **2.8** the functions $L(x)$, $M(y)$ are finite and continuous on the closed sets E_x, E_y respectively. We can define therefore

$$L = \max_{x \in E_x} L(x), \qquad M = \max_{y \in E_y} M(y).$$

3. Auxiliary rectangle functions.

3.1. We consider the general term of the sequence of subdivisions $X_\nu Y_\nu$ (see **2.5** to **2.11**) and a generic rectangle $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ of $X_\nu Y_\nu$. The reader is requested to consult **2** whenever in doubt about the meaning of the various symbols used.

3.2. We shall associate with the rectangle

$$(7) \qquad r: \quad x' \leqq x \leqq x'', \qquad y' \leqq y \leqq y''$$

the following four points of the surface (1).

$$\begin{aligned} A&: \quad x = x', \quad y = y', \quad z = f(x', y'), \\ B&: \quad x = x', \quad y = y'', \quad z = f(x', y''), \\ C&: \quad x = x'', \quad y = y'', \quad z = f(x'', y''), \\ D&: \quad x = x'', \quad y = y', \quad z = f(x'', y'). \end{aligned}$$

The symbols ABC, ACD will denote the areas, taken with the *positive* sign, of the triangles with the vertices A, B, C and A, C, D respectively. The orthogonal projections of the points A, B, C, D upon the planes xy, yz, zx respectively will be denoted by A_i, B_i, C_i, D_i, $i = 1, 2, 3$. The areas of the triangles $A_iB_iC_i$, $A_iC_iD_i$, $i = 1, 2, 3$, each taken with a *convenient* sign to be explained in a moment, will be denoted by $A_iB_iC_i$, $A_iC_iD_i$. Whatever signs we shall select for these areas, we shall have

$$(8) \qquad ABC = \| \mathfrak{x}_1 \|, \qquad ACD = \| \mathfrak{x}_2 \|,$$

where $\mathfrak{x}_1$, $\mathfrak{x}_2$ are vectors, defined by their components as follows.

$$(9) \qquad \begin{aligned} \mathfrak{x}_1 &= (A_1B_1C_1, A_2B_2C_2, A_3B_3C_3), \\ \mathfrak{x}_2 &= (A_1C_1D_1, A_2C_2D_2, A_3C_3D_3). \end{aligned}$$

The quantity

$$P_\nu = X_\nu Y_\nu (ABC + ACD)$$

represents then the area, in the elementary sense, of a polyhedron Π_ν inscribed in the surface S. By condition (a) in **2.5**, the sequence Π_ν converges to the

surface (1) in the Fréchet sense, and hence we have, by the definition of $A^*(S)$ in **1.2**,

$$\lim_{\nu\to\infty} P_\nu \geqq A^*(S) \geqq A(S).$$

3.3. We make the following agreements concerning the signs of $A_iB_iC_i$, $A_iC_iD_i$.

$$A_1B_1C_1 = \tfrac{1}{2}(x''-x')(y''-y'), \qquad A_1C_1D_1 = \tfrac{1}{2}(x''-x')(y''-y'),$$
$$A_2B_2C_2 = \tfrac{1}{2}(y''-y')(f(C_1)-f(B_1)), \qquad A_2C_2D_2 = \tfrac{1}{2}(y''-y')(f(D_1)-f(A_1)),$$
$$A_3B_3C_3 = \tfrac{1}{2}(x''-x')(f(B_1)-f(A_1)), \qquad A_3C_3D_3 = \tfrac{1}{2}(x''-x')(f(C_1)-f(D_1)).$$

We use, for brevity, $f(A_1)$ instead of $f(x', y')$ and so on.

3.4. We define

$$\alpha = (x''-x')(y''-y'),$$
$$b = \tfrac{1}{2}(y''-y')(f(D_1)+f(C_1)-f(B_1)-f(A_1)),$$
$$c = \tfrac{1}{2}(x''-x')(f(B_1)+f(C_1)-f(A_1)-f(D_1)),$$
$$n = (\alpha^2+b^2+c^2)^{1/2}.$$

If (α, b, c) denotes the vector with components α, b, c, then we have (cf. **3.2**, **3.3**)

$$(\alpha, b, c) = \mathfrak{x}_1 + \mathfrak{x}_2, \qquad n = \| \mathfrak{x}_1 + \mathfrak{x}_2 \|.^{12}$$

3.5. The inscribed polyhedron Π_ν (see **3.2**) can be represented by an equation of the form

$$\Pi_\nu: \quad z = \phi_\nu(x, y), \quad (x, y) \in I_0.$$

We have then, for example, for $x' \leqq x \leqq x''$, $y = y''$ the formula

$$\phi_\nu(x, y'') = f(B_1) + \frac{f(C_1)-f(B_1)}{x''-x'}(x-x').$$

Similarly, $\phi_\nu(x, y)$ is a linear function on each of the sides of the rectangle (7) of $X_\nu Y_\nu$. From this remark we obtain the formulas

$$b = \int_{y'}^{y''} (\phi_\nu(x'', y) - \phi_\nu(x', y))\,dy,$$
$$c = \int_{x'}^{x''} (\phi_\nu(x, y'') - \phi_\nu(x, y'))\,dx.$$

3.6. We define

$$\beta = \int_{y'}^{y''} (f(x'', y) - f(x', y))\,dy,$$
$$\gamma = \int_{x'}^{x''} (f(x, y'') - f(x, y'))\,dx,$$
$$\tau = (\alpha^2+\beta^2+\gamma^2)^{1/2}.$$

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[12] $\| \mathfrak{x} \|$ designates the length of the vector $\mathfrak{x}$.

3.7. In defining β, γ we deviate from Geöcze [13] who works with

$$\bar{\beta} = \int_{y'}^{y''} |f(x'', y) - f(x', y)|\, dy,$$

$$\bar{\gamma} = \int_{x'}^{x''} |f(x, y'') - f(x, y')|\, dx,$$

$$\bar{\tau} = (\alpha^2 + \bar{\beta}^2 + \bar{\gamma}^2)^{1/2}.$$

That is, we omit the absolute value signs appearing in the Geöcze expressions $\bar{\beta}$, $\bar{\gamma}$. The advantages of this modification were noticed in previous literature (cf. the last footnote on p. 626 in Radó [3]; see also Morrey [1]). The quantities β, γ will permit us to dispense with a considerable amount of most ingenious but quite cumbersome geometrical detail in the method of Geöcze. We have to satisfy ourselves, however, that certain important properties of $\bar{\beta}$, $\bar{\gamma}$, $\bar{\tau}$ are shared by β, γ, τ.

3.8. We shall need the following facts.

(a) $\operatorname{Lim} XY\tau$ exists, for every sequence of subdivisions XY of I_0, provided only that $\| X \| \to 0$, $\| Y \| \to 0$, and the limit is independent of the choice of the sequence XY.

(b) This limit, to be denoted by T, is $\leqq A(S)$.[14]

(c) These statements remain valid for sequences of subdivisions of any sub-rectangle $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$. The corresponding limit will be denoted by t.

(d) For every sub-rectangle we have $\tau \leqq t$.

(e) $XYt = T$ for every subdivision XY of I_0.

(f) As a direct consequence of the preceding statements, we have $XY\tau \leqq T$ for every subdivision XY of I_0, and $XY(t - \tau) \to 0$ for every sequence of subdivisions XY such that $\| X \| \to 0$, $\| Y \| \to 0$.

3.9. The proofs of these statements are precisely the same as in the case of the original Geöcze expressions $\bar{\beta}$, $\bar{\gamma}$, $\bar{\tau}$. The reader is therefore referred to the excellent presentation in Saks [1], Chapter V, especially Theorem 3.9, p. 168, or to Morrey [1].

3.10. We define, for the rectangle (7),

$$H = f(A_1) - f(B_1) + f(C_1) - f(D_1). \tag{10}$$

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[13] Geöcze [1].

[14] In fact, $T = A(S)$ for every continuous surface (1). This theorem was surmised by Geöcze who proved it however only in special cases. The general theorem was proved by the author (Radó [2]). See also the excellent presentation in Saks [1]. These authors used the original Geöcze quantities, but the proofs remain valid, with obvious modifications, for the modified quantities β, γ, τ.

Further, $s_{x'}$, $s_{x''}$, $s_{y'}$, $s_{y''}$ will denote the lengths of the arcs

$$\begin{aligned} &x = x', \quad y' \leqq y \leqq y'', \quad z = f(x', y), \\ &x = x'', \quad y' \leqq y \leqq y'', \quad z = f(x'', y), \\ &y = y', \quad x' \leqq x \leqq x'', \quad z = f(x, y'), \\ &y = y'', \quad x' \leqq x \leqq x'', \quad z = f(x, y''). \end{aligned}$$

$\sigma_{x'}$, $\sigma_{x''}$, $\sigma_{y'}$, $\sigma_{y''}$ will denote the lengths of the chords of these arcs. We also define

$$\theta_{x'} = \int_{y'}^{y''} |f(x', y) - \phi_\nu(x', y)|\, dy, \quad \theta_{x''} = \int_{y'}^{y''} |f(x'', y) - \phi_\nu(x'', y)|\, dy,$$

$$\theta_{y'} = \int_{x'}^{x''} |f(x, y') - \phi_\nu(x, y')|\, dx, \quad \theta_{y''} = \int_{x'}^{x''} |f(x, y'') - \phi_\nu(x, y'')|\, dx.$$

3.11. Put $\xi = \frac{1}{2}(x' + x'')$. We define then

$$\rho = \int_{x'}^{\xi} (\phi_\nu(x, y') - f(x, y'))\, dx + \int_{\xi}^{x''} (f(x, y') - \phi_\nu(x, y'))\, dx +$$
$$+ \int_{x'}^{\xi} (f(x, y'') - \phi_\nu(x, y''))\, dx + \int_{\xi}^{x''} (\phi_\nu(x, y'') - f(x, y''))\, dx.$$

3.12. From (10), **3.3**, (9) we obtain the identity

$$\mathfrak{x}_1 - \mathfrak{x}_2 = (0, \tfrac{1}{2}(y'' - y')H, -\tfrac{1}{2}(x'' - x')H).$$

3.13. We subdivide now the rectangle (7) into two rectangles by the line $x = \xi = \frac{1}{2}(x' + x'')$. Let us use the superscript 1 to refer to the rectangle $x' \leqq x \leqq \xi$, $y' \leqq y \leqq y''$ and the superscript 2 to refer to the rectangle $\xi \leqq x \leqq x''$, $y' \leqq y \leqq y''$. We obtain then from **3.6**, (10), **3.11** the identity

$$H = \frac{4}{x'' - x'} [\gamma^{(2)} - \gamma^{(1)} + \rho].$$

3.14. Since $A(S) < \infty$ by assumption, the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ exist a. e. in I_0 and are summable there (see Saks [1], p. 181). By the absolute continuity of the indefinite Lebesgue integral we have therefore, for $\psi > 0$, a function $\Omega(\psi)$ such that the following holds.

(a) $\Omega(\psi) \to 0$ for $\psi \to 0$.

(b) If $r_1, r_2, \cdots, r_j, \cdots$ is any system of non-overlapping rectangles in I_0 with total area $\leqq \psi$, then

$$\sum_j \iint_{r_j} |\partial f/\partial x|\, dxdy \leqq \Omega(\psi),$$

$$\sum_j \iint_{r_j} |\partial f/\partial y|\, dxdy \leqq \Omega(\psi),$$

4. Inequalities.

4.1. If $\mathfrak{a}_1$, $\mathfrak{a}_2$ are vectors, in Euclidean three-space, then it follows from the definition of $\mathfrak{a}_1 + \mathfrak{a}_2$ that

$$u = \| \mathfrak{a}_1 \|, \quad v = \| \mathfrak{a}_2 \|, \quad w = \| \mathfrak{a}_1 + \mathfrak{a}_2 \|$$

are the lengths of the sides of a triangle whose area we shall denote by Δ.

4.2. From the preceding remark the following inequalities can be derived immediately (the first three being, of course, well known).

$$\| \mathfrak{a}_1 + \mathfrak{a}_2 \| \leqq \| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \|,$$

$$(11) \qquad | \| \mathfrak{a}_1 \| - \| \mathfrak{a}_2 \| | \leqq \| \mathfrak{a}_1 - \mathfrak{a}_2 \|,$$

$$(12) \qquad \| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \| \leqq \| \mathfrak{a}_1 + \mathfrak{a}_2 \| + \| \mathfrak{a}_1 - \mathfrak{a}_2 \|,$$

$$(13) \qquad \| \mathfrak{a}_1 \times \mathfrak{a}_2 \| \leqq 2[\| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \|]^{3/2}[\| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \| - \| \mathfrak{a}_1 + \mathfrak{a}_2 \|]^{1/2}.$$

In the last inequality, $\mathfrak{a}_1 \times \mathfrak{a}_2$ designates the vector product of $\mathfrak{a}_1$, $\mathfrak{a}_2$.

4.3. The proof of (13) is as follows. By a well-known elementary formula we have

$$\Delta^2 = \mu(\mu - u)(\mu - v)(\mu - w),$$

where $\mu = (u + v + w)/2$. Thus $\mu \leqq u + v$. Hence

$$\Delta^2 \leqq \mu^3(u + v - w) \leqq (u + v)^3(u + v - w),$$

$$(14) \qquad \Delta \leqq (u + v)^{3/2}(u + v - w)^{1/2}.$$

Since $\| \mathfrak{a}_1 \times \mathfrak{a}_2 \| = 2\Delta$, (13) is proved.

4.4. Let there be given, in $x_1 \leqq x \leqq x_2$, a continuous function $f(x)$. We assume that the length s of the arc

$$(15) \qquad x_1 \leqq x \leqq x_2, \qquad y = f(x)$$

is finite. We denote by σ the length of the chord of this arc. Let

$$(16) \qquad x_1 \leqq x \leqq x_2, \qquad y = \phi(x)$$

be the equation of the chord. We assert the inequalities

$$(17) \qquad \int_{x_1}^{x_2} | f(x) - \phi(x) | \, dx \leqq s(x_2 - x_1),$$

$$(18) \qquad \int_{x_1}^{x_2} | f(x) - \phi(x) | \, dx \leqq 2s^{3/2}(s - \sigma)^{1/2}.$$

If we drop the assumption that s is finite we have the inequality

$$\int_{x_1}^{x_2} |f(x) - \phi(x)|\, dx \leqq 2(x_2 - x_1)d, \tag{19}$$

where d is the oscillation of $f(x)$ in $x_1 \leqq x \leqq x_2$.

Proof. (19) is obvious. (17) and (18) can be derived either geometrically, by enclosing the area between the arc (15) and its chord (16) in a parallelogram and in an ellipse respectively, or else by the following more analytic argument. Let F denote the maximum of $|f(x) - \phi(x)|$ in $x_1 \leqq x \leqq x_2$. Then

$$\int_{x_1}^{x_2} |f(x) - \phi(x)|\, dx \leqq (x_2 - x_1)F, \tag{20}$$

and we have a ξ, $x_1 < \xi < x_2$, such that

$$|f(\xi) - \phi(\xi)| = F. \tag{21}$$

Let us denote by P, Q, R, S the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(\xi, f(\xi))$, $(\xi, \phi(\xi))$. From the triangle PQR we see that $F = \overline{RS}$ cannot exceed $\overline{PR} + \overline{RQ}$. Hence

$$F \leqq \overline{PR} + \overline{RQ} \leqq s. \tag{22}$$

(20), (21), (22) prove (17). Next, observe that $(x_2 - x_1)F$ is twice the area of the triangle PQR. Hence, by (14),

$$(x_2 - x_1)F \leqq 2(\overline{PR} + \overline{RQ})^{5/2}(\overline{PR} + \overline{RQ} - \overline{PQ})^{1/2}. \tag{23}$$

But $\overline{PR} + \overline{RQ} \leqq s$, $\overline{PQ} = \sigma$. Hence (18) follows from (20) and (23).

We shall list now, for easier reference, a number of inequalities for the auxiliary quantities defined in **3.** The cross-references indicate the sections containing the information needed in the proofs.

4.5. $\| \mathfrak{x}_1 - \mathfrak{x}_2 \| = O(|H|/q_v)$. See (9) in **3.2**, (10) in **3.10**, **3.12**, **2.10**.

4.6. $|n - \tau| \leqq |b - \beta| + |c - \gamma|$. See **3.4**, **3.6**, (11) in **4.2**.

4.7. $|c - \gamma| \leqq \theta_{y'} + \theta_{y''}$, $|b - \beta| \leqq \theta_{x'} + \theta_{x''}$. See **3.4**, **3.5**, **3.6**, **3.10**.

4.8. We have the relations

$$\theta_{x'} = O(s_{x'}/q_v), \tag{24}$$

$$\theta_{x'} = O[s_{x'}^{3/2}(s_{x'} - \sigma_{x'})^{1/2}], \tag{25}$$

and analogous relations for $\theta_{x''}$, $\theta_{y'}$, $\theta_{y''}$. See **3.10**, **4.4**, **2.10**.

4.9. The preceding estimates for $\theta_{x'}$ may be meaningless if $x' = 0$, because then possibly $s_{x'} = \infty$. For $x' = 0$ we have the estimate

$$\theta_{x'} = O(d_\nu/q_\nu)$$

with analogous statements for $\theta_{x''}$, $\theta_{y'}$, $\theta_{y''}$. See **3.10**, (19) in **4.4**, **2.10**, **2.12**.

4.10. Let us return now to **3.13**. We shall use the superscripts 1, 2 in the sense explained there. We introduce the vectors (see **3.6**, **3.13**)

$$\mathfrak{y}^{(1)} = (\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}), \qquad \mathfrak{y}^{(2)} = (\alpha^{(2)}, \beta^{(2)}, \gamma^{(2)}).$$

Then (see **3.6**, **3.13**)

(26) $$\tau = \| \mathfrak{y}^{(1)} + \mathfrak{y}^{(2)} \|.$$

By the definition of the vector product we have

(27) $$| \alpha^{(1)}\gamma^{(2)} - \alpha^{(2)}\gamma^{(1)} | \leqq \| \mathfrak{y}^{(1)} \times \mathfrak{y}^{(2)} \|.$$

By **3.13** we have

(28) $$\alpha^{(1)} = \alpha^{(2)} = \tfrac{1}{2}\alpha.$$

(27) and (28) yield

(29) $$| \gamma^{(2)} - \gamma^{(1)} | \leqq 2/\alpha \, \| \mathfrak{y}^{(1)} \times \mathfrak{y}^{(2)} \|.$$

By **3.13**, **3.6**, **3.8**, (26) we have

(30) $$\| \mathfrak{y}^{(1)} \| + \| \mathfrak{y}^{(2)} \| = \tau^{(1)} + \tau^{(2)} \leqq t^{(1)} + t^{(2)} = t,$$

(31) $$\| \mathfrak{y}^{(1)} \| + \| \mathfrak{y}^{(2)} \| - \| \mathfrak{y}^{(1)} + \mathfrak{y}^{(2)} \| = \tau^{(1)} + \tau^{(2)} - \tau \leqq t - \tau.$$

(29), (13) in **4.2**, (30), (31), **3.4**, **2.10** yield now

(32) $$| \gamma^{(2)} - \gamma^{(1)} | = O[q_\nu^2 t^{3/2}(t-\tau)^{1/2}].$$

4.11. $| \rho | \leqq \theta_{y'} + \theta_{y''}$. See **3.10**.

4.12. $| H | = O[q_\nu^3 t^{3/2}(t-\tau)^{1/2} + q_\nu(\theta_{y'} + \theta_{y''})]$. See **3.13**, (32) in **4.10**, **4.11**.

4.13. $ABC + ACD - n = O[q_\nu^2 t^{3/2}(t-\tau)^{1/2} + \theta_{y'} + \theta_{y''}]$.
See (8) in **3.2**, **3.4**, (12) in **4.2**, **4.5**, **4.12**.

4.14. From the definition of H (see **3.10**) it is obvious that

$$| H | \leqq s_{y'} + s_{y''}.$$

Hence (see references in **4.13**)

$$ABC + ACD - n = O\left(\frac{s_{y'} + s_{y''}}{q_\nu}\right).$$

4.15. This last estimate may become meaningless if $y' = 0$ or $y'' = 1$, since then possibly $s_{y'} = \infty$ or $s_{y''} = \infty$. We derive therefore an alternative estimate. By **3.10** and **2.12** it is obvious that

$$|H| = O(d_\nu).$$

By the same cross-references as in **4.14** we obtain then the estimate

$$ABC + ACD - n = O(d_\nu/q_\nu).$$

4.16. Let there be given now a positive constant K. For each rectangle r of $X_\nu Y_\nu$ we define then a quantity λ_K as follows. $\lambda_K = 1$ if for the rectangle r *all* the inequalities

$$s_{x'},\ s_{x''},\ s_{y'},\ s_{y''} \leqq K/q_\nu, \qquad t \leqq K/q_\nu^2$$

hold. Otherwise $\lambda_K = 0$ for that rectangle. We propose to show that

$$X_\nu Y_\nu (1 - \lambda_K)\alpha = O(1/K + 1/q_\nu), \tag{33}$$

where α denotes the area of the generic rectangle r of $X_\nu Y_\nu$.

4.17. Let us consider first those rectangles of $X_\nu Y_\nu$ for which

$$s_{x'} > K/q_\nu \quad \text{and} \quad x' \neq 0. \tag{34}$$

Let $N(x')$ designate the number of such rectangles for fixed x'. Then (see **1.4**, **2.10**, **2.4**, **2.5**)

$$\frac{N(x')K}{q_\nu} \leqq L(x'),$$

$$N(x') \leqq \frac{q_\nu^2}{K}\ \frac{L(x')}{q_\nu} = O[(q_\nu^2/K)L(x')(x'' - x')],$$

$$X^*_\nu N(x') = O[(q_\nu^2/K)X^*_\nu L(x')(x'' - x')]$$
$$= O[(q_\nu^2/K)\left(\int_0^1 L(x)dx + \epsilon\right)] = O(q_\nu^2/K).$$

Thus the number of all rectangles satisfying (34) is $O(q_\nu^2/K)$ and hence their total area is $O(1/K)$. The sum of the areas of those rectangles of $X_\nu Y_\nu$ for which $x' = 0$ is $O(1/q_\nu)$ (cf. **2.10**). Hence, the sum of the areas of those rectangles of $X_\nu Y_\nu$ for which $s_{x'} > K/q_\nu$ is $O(1/K + 1/q_\nu)$. Applying the same reasoning to $s_{x''}$, $s_{y'}$, $s_{y''}$, it follows that the sum of the areas of those rectangles of $X_\nu Y_\nu$ for which the inequalities

$$s_{x'},\ s_{x''},\ s_{y'},\ s_{y''} \leqq K/q_\nu$$

do *not* hold simultaneously, is $O(1/K + 1/q_\nu)$.

4.18. Let now N designate the number of those rectangles of $X_\nu Y_\nu$ for which $t > K/q_\nu^2$. Then (see **3.8**)

$$NK/q_\nu^2 \leqq T,$$

and hence

$$N = O(q_\nu^2/K).$$

Thus the sum of the areas of these rectangles is $O(1/K)$. Hence (see **4.16**, **4.17**)

$$X_\nu Y_\nu (1 - \lambda_K)\alpha = O(1/K + 1/q_\nu) + O(1/K) = O(1/K + 1/q_\nu),$$

and (33) is proved.

4.19. $\underset{x' \epsilon e_x}{X^*_\nu Y_\nu s_{x'}/q_\nu}$, $\underset{x'' \epsilon e_x}{X^*_\nu Y_\nu s_{x''}/q_\nu}$, $\underset{y' \epsilon e_y}{X_\nu Y^*_\nu s_{y'}/q_\nu}$, $\underset{y'' \epsilon e_y}{X_\nu Y^*_\nu s_{y''}/q_\nu} = O(\epsilon)$.

Let us consider the first summation. By **2.4** to **2.10**

$$\underset{x' \epsilon e_x}{X^*_\nu Y_\nu s_{x'}/q_\nu} = O[\underset{x' \epsilon e_x}{X^*_\nu Y_\nu s_{x'}}(x'' - x')] = O[\underset{x' \epsilon e_x}{X^*_\nu L(x')}(x'' - x')] = O(\epsilon).$$

4.20. We have

$$(35) \qquad \underset{\substack{y' \epsilon E_y \\ s_{y'} > K^{1/2}/q_\nu}}{X_\nu Y_\nu s_{y'}/q_\nu}, \qquad \underset{\substack{y'' \epsilon E_y \\ s_{y''} > K^{1/2}/q_\nu}}{X_\nu Y_\nu s_{y''}/q_\nu} = O[\omega(pM/K^{1/2})].$$

Proof. It is sufficient to consider the first summation (see **2.6**, **2.8**, **2.5**, **2.13**, **4.16** for notations). Let $N(y')$ designate the number of arcs $s_{y'}$, for fixed y', that contribute to the first summation. Then by **2.13**

$$\frac{N(y')K^{1/2}}{q_\nu} \leqq M, \qquad N(y') \leqq \frac{Mq_\nu}{K^{1/2}}.$$

For fixed y', the sum of the intervals $x' \leqq x \leqq x''$ corresponding to these arcs is therefore (see **2.5**, (c))

$$\leqq N(y')p/q_\nu \leqq pM/K^{1/2}.$$

By **2.8**, (4) it follows that, for fixed y', the sum of the lengths of these arcs $s_{y'}$ is $\leqq \omega(pM/K^{1/2})$. Since y' takes on not more than q_ν values, (35) is established.

4.21. $X_\nu Y^*_\nu (1 - \lambda_K) s_{y'}/q_\nu$, $X_\nu Y^*_\nu (1 - \lambda_K) s_{y''}/q_\nu$
$$= O[\epsilon + \omega(pM/K^{1/2}) + 1/K^{1/2} + K^{1/2}/q_\nu].$$

Proof. For notations see **2.4**, **4.16**, **2.5**, **2.8**, **2.13**. It is sufficient to consider the first summation. We split it into three summations Σ_1, Σ_2, Σ_3 as follows. Σ_1 is relative to those terms for which $y' \epsilon e_y$ (see **2.7**). Σ_2 is relative to those terms for which

$$y' \in E_y \quad \text{and} \quad s_{y'} > K^{1/2}/q_\nu.$$

Σ_3 is relative to those terms for which

$$y' \in E_y \quad \text{and} \quad s_{y'} \leqq K^{1/2}/q_\nu.$$

By **4.19**, $\Sigma_1 = \theta(\epsilon)$. By **4.20**, $\Sigma_2 = \theta(\omega(pM/K^{1/2}))$. Finally, by **2.10**, **4.16**,

$$\Sigma_3 \leqq X_\nu Y_\nu (1-\lambda_K) K^{1/2}/q_\nu^2 = K^{1/2} O[X_\nu Y_\nu (1-\lambda_K)\alpha]$$
$$= K^{1/2} O[1/K + 1/q_\nu] = O[1/K^{1/2} + K^{1/2}/q_\nu].$$

4.22. $\underset{x' \in e_x}{X_\nu Y_\nu \theta_{x'}}, \underset{x'' \in e_x}{X_\nu Y_\nu \theta_{x''}}, \underset{y' \in e_y}{X_\nu Y_\nu \theta_{y'}}, \underset{y'' \in e_y}{X_\nu Y_\nu \theta_{y''}} = O(\epsilon + d_\nu)$.

See (24) in **4.8**, **4.19**, **4.9**.

4.23. $\underset{s_{x'} \leqq K/q_\nu}{X_\nu Y_\nu \theta_{x'}}, \underset{s_{x''} \leqq K/q_\nu}{X_\nu Y_\nu \theta_{x''}}, \underset{s_{y'} \leqq K^{1/2}/q_\nu}{X_\nu Y_\nu \theta_{y'}}, \underset{s_{y''} \leqq K^{1/2}/q_\nu}{X_\nu Y_\nu \theta_{y''}} = O(\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2})$.

Proof. It is sufficient to consider the first summation. We split it into two summations Σ_1, Σ_2, where Σ_1 is relative to those terms for which $x' \in e_x$, and Σ_2 is relative to those terms for which $x' \in E_x$. By **4.22**, $\Sigma_1 = O(\epsilon + d_\nu)$. By (25) in **4.8**, the inequality of Schwartz, and (6) in **2.11**,

$$\Sigma_2 = O[\underset{\substack{x' \in E_x \\ s_{x'} \leqq K/q_\nu}}{X_\nu Y_\nu s_{x'}^{3/2}} (s_{x'} - \sigma_{x'})^{1/2}] = O[(K^{3/2}/q_\nu^{3/2}) \underset{x' \in E_x}{X_\nu} (Y_\nu 1^{1/2} (s_{x'} - \sigma_{x'})^{1/2})]$$
$$= O[(K^{3/2}/q_\nu^{3/2}) \underset{x' \in E_x}{X_\nu} q_\nu^{1/2} (L(x') - l_\nu(x'))^{1/2}]$$
$$= O[K^{3/2}/q_\nu^{3/2} \cdot q_\nu^{1/2} \cdot q_\nu \delta_\nu^{1/2}] = O(K^{3/2}\delta_\nu^{1/2}].$$

4.24. We have

(36) $X_\nu Y_\nu \lambda_K \mid ABC + ACD - \tau \mid = O[\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2} + K^{3/2}(T - X_\nu Y_\nu \tau)^{1/2}]$.

Proof. We start with

$$\mid ABC + ACD - \tau \mid \leqq \mid ABC + ACD - n \mid + \mid n - \tau \mid.$$

By **4.6**, **4.7**, **4.23**

(37) $$X_\nu Y_\nu \lambda_K \mid n - \tau \mid = O[\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2}].$$

By **4.13**, **4.23**

(38) $X_\nu Y_\nu \lambda_K \mid ABC + ACD - n \mid$
$$= O[X_\nu Y_\nu \lambda_K q_\nu^2 t^{3/2} (t-\tau)^{1/2}] + O[\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2}].$$

By **4.16**, **3.8**, and the inequality of Schwartz,

(39) $X_\nu Y_\nu \lambda_K q_\nu^2 t^{3/2} (t-\tau)^{1/2} \leqq X_\nu Y_\nu \lambda_K q_\nu^2 (K^{3/2}/q_\nu^3)(t-\tau)^{1/2}$
$$\leqq (K^{3/2}/q_\nu) X_\nu Y_\nu 1^{1/2} (t-\tau)^{1/2}$$
$$\leqq (K^{3/2}/q_\nu)(q_\nu^2)^{1/2} [X_\nu Y_\nu (t-\tau)]^{1/2} = K^{3/2}(T - X_\nu Y_\nu \tau)^{1/2}.$$

(37), (38), (39) imply (36).

4.25. We have

$$(40)\quad X_\nu Y_\nu(1-\lambda_K)\,|\,ABC+ACD-\tau\,| = O\{\Omega(1/K+1/q_\nu)+d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu\}.$$

Proof. By **4.6**, **4.7**

$$(41)\quad X_\nu Y_\nu(1-\lambda_K)\,|\,n-\tau\,| \leqq X_\nu Y_\nu(1-\lambda_K)(|\,\beta\,|+|\,b\,|+\theta_{y'}+\theta_{y''}).$$

By (33) in **4.16**

$$X_\nu Y_\nu(1-\lambda_K)\alpha = O(1/K+1/q_\nu),$$

and hence, by **3.14**, **3.6**, **2.1**,

$$(42)\quad X_\nu Y_\nu(1-\lambda_K)\,|\,\beta\,| \leqq \Omega[O(1/K+1/q_\nu)].$$

From the definition of b (see **3.4**) it is clear that

$$|\,b\,| \leqq \tfrac{1}{2}(y''-y')(s_{y'}+s_{y''}) = O(s_{y'}/q_\nu + s_{y''}/q_\nu),$$

and (see **2.12**)

$$|\,b\,| \leqq (y''-y')d_\nu = O(d_\nu/q_\nu).$$

Hence, by **2.4**, **4.21**,

$$(43)\quad X_\nu Y_\nu(1-\lambda_K)\,|\,b\,| = O(d_\nu)+O[X_\nu Y^*_\nu(1-\lambda_K)(s_{y'}/q_\nu+s_{y''}/q_\nu)] = O[d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu].$$

By **4.8**, **4.9**, **4.21**,

$$(44)\quad X_\nu Y_\nu(1-\lambda_K)(\theta_{y'}+\theta_{y''}) = O(d_\nu)+O[X_\nu Y^*_\nu(1-\lambda_K)(s_{y'}/q_\nu+s_{y''}/q_\nu)] = O[d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu]$$

By (41), (42), (43), (44),

$$(45)\quad X_\nu Y_\nu(1-\lambda_K)\,|\,n-\tau\,| = O\{\Omega[O(1/K+1/q_\nu)] + d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu\}.$$

By **4.14**, **4.15**, **4.21** we obtain similarly

$$(46)\quad X_\nu Y_\nu(1-\lambda_K)\,|\,ABC+ACD-n\,| = O[d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu].$$

(45), (46) imply (40).

4.26. As a direct consequence of **4.24** and **4.25** we obtain

$$X_\nu Y_\nu\,|\,ABC+ACD-\tau\,| = O\{\Omega[O(1/K+1/q_\nu)]+\omega(pM/K^{1/2})+\epsilon+\delta_\nu+1/K^{1/2}+K^{1/2}/q_\nu + K^{3/2}\delta_\nu^{1/2}+K^{3/2}(T-X_\nu Y_\nu\tau)^{1/2}\}.$$

5. Proof of the theorem.

5.1. LEMMA. *For the sequence $X_\nu Y_\nu$, described in* **2.5** *to* **2.9** *we have*

$$\overline{\lim_{\nu\to\infty}} X_\nu Y_\nu \mid ABC + ACD - \tau \mid \leqq \Gamma\epsilon, \tag{47}$$

where Γ is a constant that depends only *upon the given surface* (1) *and the constant p of* **2.5**.

Proof. In **4.26** keep K fixed and make $\nu \to \infty$. By **2.5**, **2.12**, (5) in **2.11**, (e) and (f) in **3.8** we obtain

$$\overline{\lim_{\nu\to\infty}} X_\nu Y_\nu \mid ABC + ACD - \tau \mid = O\{\Omega[O(1/K)] + \omega(pM/K^{1/2}) + \epsilon + 1/K^{1/2}\}.$$

This relation holds for every $K > 0$. Hence, making $K \to \infty$, we obtain, by **3.14**, **2.8**,

$$\overline{\lim_{\nu\to\infty}} X_\nu Y_\nu \mid ABC + ACD - \tau \mid = O(\epsilon).$$

By the meaning of the symbol O (see **2.9**), this last relation proves (47).

5.2. By **3.2**, we have

$$X_\nu Y_\nu (ABC + ACD) = E(\Pi_\nu),$$

where $E(\Pi_\nu)$ is the elementary area of an inscribed polyhedron

$$\Pi_\nu: \quad z = \phi_\nu(x, y), \quad (x, y) \in I_0,$$

such that $\phi_\nu(x, y) \to f(x, y)$ uniformly in I_0. Since (cf. **3.8**)

$$\begin{aligned} \mid E(\Pi_\nu) - T \mid &= \mid X_\nu Y_\nu (ABC + ACD) - X_\nu Y_\nu \tau + X_\nu Y_\nu (\tau - t) \mid \\ &\leqq X_\nu Y_\nu \mid ABC + ACD - \tau \mid + X_\nu Y_\nu (t - \tau) \end{aligned}$$

the relation (47) implies that (cf. **3.8**, (f))

$$\overline{\lim_{\nu\to\infty}} \mid E(\Pi_\nu) - T \mid \leqq \Gamma\epsilon. \tag{48}$$

On the other hand, as noted above,

$$\max_{(x,y)\in I_0} \mid f(x, y) - \phi_\nu(x, y) \mid \to 0. \tag{49}$$

We recall that ϵ, p were assigned arbitrarily, and that Γ in (48) is *independent* of ϵ and of the sequence $X_\nu Y_\nu$. Hence, for any positive integer j, we can select first p to satisfy

$$1 < p < 1 + 1/j.$$

Once p has been so selected, the constant Γ in (48) is determined. Then we select $\epsilon > 0$ to satisfy

$$\Gamma\epsilon < 1/j.$$

After ϵ has been so selected, we have a sequence $X_\nu Y_\nu$ as described in **2.5** to **2.9**, and (48) and (49) hold. For ν large enough, we have then, by (48) and (49),

$$(50) \qquad |E(\Pi_\nu) - T| < 1/j, \qquad \max_{(x,y)\,\epsilon\, I_0} |f(x,y) - \phi_\nu(x,y)| < 1/j.$$

We take such a ν, and we designate the corresponding X_ν, Y_ν, ϕ_ν, Π_ν by $X^{(j)}$, $Y^{(j)}$, $\phi^{(j)}$, $\Pi^{(j)}$ to stress their dependence upon the positive integer j. We obtain in this manner the following lemma.

5.3. LEMMA. *We have a sequence of subdivisions $X^{(j)}Y^{(j)}$, and a corresponding sequence of inscribed polyhedra*

$$\Pi^{(j)}: \quad z = \phi^{(j)}(x,y), \quad (x,y)\,\epsilon\, I_0,$$

such that the following holds (cf. **2.5** *and* (50)).

(a) $\phi^{(j)}(x,y) \underset{j\to\infty}{\to} f(x,y)$ *uniformly in* I_0.

(b) $q(X^{(j)}) = q(Y^{(j)})$, $j = 1, 2, \cdots$.

(c) $p(X^{(j)}) \to 1$, $p(Y^{(j)}) \to 1$ *for* $j \to \infty$.

(d) $E(\Pi^{(j)}) \underset{j\to\infty}{\to} T$.

5.4. The proof of the theorem of **1.5** is now immediate. By the definition of $A^*(S)$ in **1.2** we have

$$(51) \qquad \underline{\lim}\, E(\Pi^{(j)}) \geqq A^*(S).$$

By **3.8**, (b) we have

$$(52) \qquad T \leqq A(S).$$

By **1.2**

$$(53) \qquad A(S) \leqq A^*(S).$$

(51), (52), (53) and (d) in **5.3** imply $A(S) = A^*(S)$.

5.5. In fact we have proved more than the mere equality $A(S) = A^*(S)$. Indeed, we also proved that we have sequences of inscribed polyhedra as described in **5.3**. The conditions stated there express the fact that the approximating inscribed polyhedron $\Pi^{(j)}$ is such that its projection upon the xy plane consists of a number of triangles obtained by subdividing the unit

square I_0 first into rectangles that are *practically congruent squares*, and then drawing one diagonal in each of these rectangles.

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INTERPOLATED DENUMERANTS AND LAMBERT SERIES.*

By E. T. Bell.

1. Notation, summary. The letters denoting integers, $r > 1$; $a_1, \cdots, a_r$ are constant and > 0; $x_1, \cdots, x_r$ are variable and $\geqq 0$. The number $D(n|a_1, \cdots, a_r)$ of sets $(x_1, \cdots, x_r)$ of solutions of

$$a_1x_1 + \cdots + a_rx_r = n \tag{1.1}$$

is (in Sylvester's terminology) the denumerant of n with respect to $a_1, \cdots, a_r$. To avoid trivial modifications in the notation, the case $a_1 = \cdots = a_r = a$ is excluded; $D(n|a, \cdots, a)$ is the coefficient of x^n in $(1-x^a)^{-r}$, $=1$ if $n = 0$, $=0$ if $a \nmid n$, $= \dfrac{a^{1-r}}{(r-1)!}\prod_{j=1}^{r-1}(n+ja)$ if $a|n$. When there can be no ambiguity, we abbreviate $D(n|a_1, \cdots, a_r)$ to $D(n)$, and, if necessary, state the associated equation (1.1).

The L. C. M. of $a_1, \cdots, a_r$ will be denoted by a. $n_1, \cdots, n_r$ are any r different non-negative constant integers. Hence

$$D(an_j + b_j), \qquad 0 \leqq b_j < a, \qquad (j = 1, \cdots, r) \tag{1.2}$$

are constants.

In Sylvester's method,[1]

$$D(n) = A(n) + U$$

where $A(n)$ is a polynomial in n of degree $r-1$, and U, the 'undulant' part, contains roots of unity. Sylvester states[2] that

$$A(n) = \sum_{j=0}^{r-1} A_{r-1-j}n^j/j!.$$

where A_s is the coefficient of x^s in the power series expansion of $\prod_{j=1}^{r}(1-e^{-a_jx})^{-1}$. Cayley's method leads to similar conclusions, the periodic terms in $D(n)$ being collected in his 'circulators.' Our object is to get rid of both undulants

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[1] For the history of denumerants to 1919, see L. E. Dickson, *History of the Theory of Numbers*, vol. 2 (1920), chap. 3, and for an exposition of Sylvester's and Cayley's methods, either the papers in vol. 2 of each of these authors' works or P. Bachmann, *Niedere Zahlentheorie*, zweiter Teil (1910), Kap. 3. The results stated by Sylvester in 1857 (*loc. cit.*, No. 17) are repeated, and most are proved, in several later papers.

[2] *Loc. cit.*, 87, last footnote.

and circulators, as these are responsible for the most awkward part of the calculations, by suitably classifying the arguments n in $D(n)$.

It will be shown (**2**) that $D(an+b)$, for any constant integer b such that $0 \leqq b < a$, is a polynomial in n of degree $r-1$,

$$D(an+b) = c_0 + c_1 n + \cdots + c_{r-1} n^{r-1}, \tag{1.3}$$

with constant (independent of n) rational number coefficients which may be explicitly evaluated in terms of the constants (1.2) with $b_j = b$.

If in (1.3) we put $n = n_1, \cdots, n_r$, and from the resulting equations and (1.3) eliminate $c_0, \cdots, c_{r-1}$, we get

$$\begin{vmatrix} D(an+b) & 1 & n & \cdots & n^{r-1} \\ D(an_1+b) & 1 & n_1 & \cdots & n_1^{r-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ D(an_r+b) & 1 & n_r & & n_r^{r-1} \end{vmatrix} = 0.$$

The coefficients of the elements in the first column in the expanded form of the left are determinants of a familiar type (alternants), whose expansions as products of differences are immediate. Or we may apply Lagrange's interpolation formula to the polynomial in (1.3):

$$F(x) \equiv \prod_{j=1}^{r} (x - n_j), \qquad F_j(x) \equiv F(x)/(x - n_j);$$

$$D(an+b) = \sum_{j=1}^{r} \frac{F_j(n)}{F_j(n_j)}\, D(an_j+b). \tag{1.4}$$

The constants n_j are at our disposal. For the choices $n_j = j-1$, or $n_j = j$, (1.4) takes a simple form. If $n_j = j$,

$$D(an+b) = (-1)^r (n-1)(n-2)\cdots(n-r)\phi_n(a,b), \tag{1.5}$$

$$\phi_n(a,b) \equiv \sum_{j=1}^{r} \frac{(-1)^j D(ja+b)}{(j-1)!\,(r-j)!\,(n-j)},$$

which may be written

$$(-1)^r r!\, D(an+b) = (n-1)\cdots(n-r) \sum_{j=1}^{r} (-1)^j j \binom{r}{j} \frac{D(ja+b)}{n-j}. \tag{1.6}$$

It follows that the r constants c_j in (1.3) are linear homogeneous functions, with constant rational number coefficients, of the r constants $D(an_j+b)$ for (1.4), and of $D(ja+b)$ for (1.6), $j = 1, \cdots, r$. The complete set of formulas $D(n)$ for a particular equation (1.1) therefore requires the calculation of the numerical values of ra constants of the kind (1.2). This may be done directly or by the classical methods cited.[1]

It will be seen in the proof of (1.3) that b may be any non-negative integer. Hence in (1.6) we may take $n=0$ and in the result replace b by the arbitrary non-negative integer n,

$$D(n)+\sum_{j=1}^{r}(-1)^j\binom{r}{j}D(n+ja)=0. \tag{1.7}$$

For $n=0$ this enables us to calculate $D(ra)$ when $D(ja)$, $j=1,\cdots,r-1$ are known,

$$1+\sum_{j=1}^{r}(-1)^j\binom{r}{j}D(ja)=0. \tag{1.8}$$

From (1.6) we have

$$D(an+b)=O(n^{r-1}), \tag{1.9}$$

the constant implied in the order function being

$$\frac{(-1)^r}{r!}\sum_{j=1}^{r}(-1)^j j\binom{r}{j}D(ja+b).$$

The enumeration of compound partitions being reducible to that of simple partitions, there are formulas corresponding to the preceding for compound denumerants.

To illustrate (1.5), let $r=3$. Then a is the L. C. M. of a_1, a_2, a_3, and $D(m)$ refers to

$$a_1x_1+a_2x_2+a_3x_3=m:$$

$$\begin{aligned}2D(an+b)=&(n-2)(n-3)D(a+b)-2(n-1)(n-3)D(2a+b)\\&+(n-1)(n-2)D(3a+b),\qquad 0\leqq b<a.\end{aligned}$$

For large values of n, this $D(an+b)$ is approximately

$$\tfrac{1}{2}n^2[D(a+b)-2D(2a+b)+D(3a+b)].$$

2. Proof of (1.3). Everything will be proved if (1.3) is. With the notation as in **1**, let α be a primitive a-th root of 1. Then if $a_j>1$, there is a least integer $s_j>0$ such that α^{s_j} is a primitive a_j-th root of 1. In the set $\alpha^{s_jt_j}$ $(t_j=1,\cdots,a_j-1;\ j=1,\cdots,r)$ let $\gamma,\delta,\cdots,\tau$ be all the unequal elements of the set, and let these occur precisely $g,d,\cdots,t$ times respectively. (If $s_jt_j\equiv s_it_i$, mod a, the corresponding elements are equal.) Then, $|x|<1$,

$$\begin{aligned}\sum_{n=0}^{\infty}D(n)x^n&=\prod_{j=1}^{r}(1-x^{a_j})^{-1},\\&=(1-x)^{-r}(x-\gamma)^{-g}(x-\delta)^{-d}\cdots(x-\tau)^{-t}.\end{aligned}$$

No $1-x^{a_j}$ has a repeated root, and there are precisely r such binomials. Hence none of $g,d,\cdots,t$ can exceed r.

The resolution of the above generating function into partial fractions may be written

$$\begin{aligned}
&R_1(1-x)^{-1}+R_2(1-x)^{-2}+\cdots+R_r(1-x)^{-r}\\
&-G_1(x-\gamma)^{-1}+G_2(x-\gamma)^{-2}+\cdots+(-1)^rG_r(x-\gamma)^{-r}\\
&-D_1(x-\delta)^{-1}+D_2(x-\delta)^{-2}+\cdots+(-1)^rD_r(x-\delta)^{-r}\\
&-\cdots\qquad\qquad\cdots\qquad\qquad\cdots\\
&-T_1(x-\tau)^{-1}+T_2(x-\tau)^{-2}+\cdots+(-1)^rT_r(x-\tau)^{-r},
\end{aligned}$$

in which the capital letters denote constants, and since $g\leqq r$, $d\leqq r,\cdots,t\leqq r$, some of the constants $G, D,\cdots, T$ may be identically zero. The coefficient of y^n in $(1-y)^{-s}$, s an integer >0, is $(n)_s$ where $(n)_1=n$, and

$$(s-1)\,!\,(n)_s\equiv(n+1)(n+2)\cdots(n+s-1),\qquad s>1.$$

If λ is any one of $\gamma, \delta,\cdots,\tau$,

$$(-1)^j(x-\lambda)^{-j}=\lambda^{-j}(1-\lambda^{-1}x)^{-j},$$

in which the coefficient of x^n is $\lambda^{-n-j}(n)_j$. Hence

$$D(n)=\sum_{j=1}^{r}(n)_j[R_j+G_j\gamma^{-n-j}+\cdots+T_j\tau^{-n-j}].$$

Since $\lambda^{mq}=1$ for all integers m,

$$D(an+b)=\sum_{j=1}^{r}(an+b)_j[R_j+G_j\gamma^{-b-j}+\cdots+T_j\tau^{-b-j}].$$

The expression in square brackets is a constant, say k_j. Hence

$$D(an+b)=\sum_{j=1}^{r}k_j(an+b)_j,$$

which may be written in the form (1.3), since $(an+b)_j$ is a polynomial of degree $j-1$ in n with coefficients independent of n.

3. An r-fold Lambert series. One possible generalization of a (1-fold) Lambert series

$$\sum_{n=1}^{\infty}\frac{f(n)x^n}{1-x^n},\qquad=\sum_{n=1}^{\infty}\Big[\sum_{d\mid n}f(d)\Big]x^n,$$

is suggested by (1.3). If in this series x is replaced by x^c, where c is a constant integer >0, the effect is merely that of a change of variable with an obvious modification of the convergence criteria.[3] It will be assumed that $f(n)$ is a numerical function of n such that all series in which it occurs here are convergent.

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[3] As stated, for example, by K. Knopp, *Journal für Mathematik*, vol. 142 (1913), pp. 283-315, especially 287, Satz 1.

With $a_1, \cdots, a_r$, a, b, $D(n)$ as in **1**, and $s \equiv a_1 + \cdots + a_r$, we define an r-fold Lambert series $L(x; a_1, \cdots, a_r; f) \equiv L(x)$ for the numerical function f and the parameters $a_1, \cdots, a_r$ by

$$L(x) \equiv \sum_{n=1}^{\infty} \frac{f(n)x^{sn}}{(1-x^{a_1 n}) \cdots (1-x^{a_r n})}.$$

To expand $L(x)$ in a power series we multiply the generating function of $D(n)$ by $f(t)x^s$, replace x by x^t in the result, and sum over all integers $t > 0$. Thus

$$L(x) = \sum_{t=1}^{\infty} [f(t)x^{st} + \sum_{n=1}^{\infty} f(t)D(n)x^{(n+s)t}],$$

$$= \sum_{t=1}^{\infty} f(t)x^{st} + \sum_{m=s+1}^{\infty} [\sum f(\delta)D(d-s)]x^m,$$

in which $\sum$ refers to all pairs (d, δ) of positive integers such that $d\delta = m$, $d > s$. If there are none such for a particular m, $\sum$ is vacuous and there is no term x^m.

We now reduce $D(d-s)$ by (1.3), noting that c_j in (1.3) depends on b, say $c_j \equiv c_j(b)$, so that

$$D(n) = \sum_{j=0}^{r-1} c_j(b)n^j.$$

For b fixed let (d_b, δ_b) be a (d, δ) such that

$$d_b \equiv s + b \bmod a, \qquad d_b > s + b, \qquad d_b\delta_b = m.$$

Then $(d_b - s - b)/a$ is an integer > 0, and by (1.3),

$$D(d_b - s) \equiv g_b(d_b) = a^{1-r} \sum_{j=0}^{r-1} c_j(b)a^{r-j-1}(d_b - s - b)^j,$$

and hence

$$\sum f(\delta)D(d-s) = \sum_{b=0}^{a-1} g_b(d_b)f(\delta_b).$$

Finally, then, the complete expansion is

$$L(x) = f(1)x^s + \sum_{m=s+1}^{\infty} [f(m/s) + \sum_{b=0}^{a-1} g_b(d_b)f(\delta_b)]x^m,$$

in which $f(y) = 0$ if y is not an integer > 0.

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ON THE BLOCH-LANDAU CONSTANT.*

By HANS RADEMACHER.

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1. In 1929 Landau, in a paper [1] on the Bloch constant, introduced besides the Bloch constant $\mathfrak{B}$ two other constants $\mathfrak{L}$ and $\mathfrak{A}$ of which only the first will interest us here. The definition of the Bloch constant is the following:

There exists a constant $\mathfrak{B}$ such that to every $\epsilon > 0$ and to every function $w = f(z)$ which is regular for $|z| < 1$ with $f'(0) = 1$ there exists a closed region $G = G_{f,\epsilon}$ in the interior of the unit-circle which is mapped by the function $w = f(z)$ in a one-to-one manner on a circle of radius $\mathfrak{B} - \epsilon$ in the w-plane; whereas there exist functions of the same specifications by which no subregion of the unit-circle is mapped in a one-to-one manner on a circle of radius $\mathfrak{B} + \epsilon$.

By dropping the condition of bi-uniqueness Landau defines another constant which we shall call the Bloch-Landau constant $\mathfrak{L}$: There exists a constant $\mathfrak{L}$ such that every function $w = f(z)$ which is regular in $|z| < 1$ with $f'(0) = 1$ assumes all values of a certain circle of radius $\mathfrak{L} - \epsilon$ in the w-plane, for any given positive ϵ; whereas to every $\epsilon > 0$ there exists a function of the same specifications whose values do not completely fill any circle of radius $\mathfrak{L} + \epsilon$.

Since every "Bloch circle" in the w-plane is a "Landau-circle," but not conversely, it is clear that we have $\mathfrak{B} \leqq \mathfrak{L}$. Landau, in the paper cited, proves also

$$0.39 < \mathfrak{B}, \qquad 0.43 \leqq \mathfrak{L} \leqq 2^{2/3}\pi\Gamma(\tfrac{1}{3})^3\Gamma(\tfrac{1}{4})^{-4},$$

where the latter upper bound is established by a certain example of $f(z)$ which does not permit a greater Landau circle. Landau gave no separate upper estimate for $\mathfrak{B}$. This was done some years ago by Ahlfors and Grunsky who proved [2] by a suitable example that

$$\mathfrak{B} \leqq (1 + 3^{\frac{1}{2}})^{\frac{1}{2}}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{3}{4})\Gamma(\tfrac{1}{12})^{-1} = 0.4719\cdots.$$

I shall show that their method leads also to an improved upper estimate of $\mathfrak{L}$.

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* Received March 21, 1942.

[1] E. Landau, "Über die Blochsche Konstante und zwei verwandte Weltkonstanten," *Mathematische Zeitschrift*, vol. 30 (1929), pp. 608-634.

[2] L. V. Ahlfors and H. Grunsky, "Über die Blochsche Konstante," *Mathematische Zeitschrift*, vol. 42 (1937), pp. 671-673.

2. The interior of the unit circle $|\zeta| < 1$ is mapped on an equilateral equiangular circular triangle of angles $\alpha\pi$ $(0 \leqq \alpha < 1)$ by means of the function

$$(1) \quad w = \phi_\alpha(\zeta) = \zeta \cdot \frac{\int_0^1 t^{-1/2-\alpha/2}(1-t)^{-1/6+\alpha/2}(1-\zeta^3 t)^{-5/6+\alpha/2}dt}{\int_0^1 t^{-1/2-\alpha/2}(1-t)^{-5/6+\alpha/2}(1-\zeta^3 t)^{-1/6+\alpha/2}dt}$$

$$= \frac{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6}+\alpha/2)}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{6}+\alpha/2)} \cdot \zeta \cdot \frac{F(\frac{5}{6}-\alpha/2, \frac{1}{2}-\alpha/2, \frac{4}{3}; \zeta^3)}{F(\frac{1}{6}-\alpha/2, \frac{1}{2}-\alpha/2, \frac{2}{3}; \zeta^3)},$$

where $F(a, b, c; z)$ is that branch of the hypergeometric function which is, for $|z| < 1$, represented by the Gaussian hypergeometric series. In the mapping established by (1) the points $1, \rho, \rho^2$ $(\rho = e^{2\pi i/3})$ of the ζ-plane correspond to the vertices of the circular triangle in the w-plane.

The formula (1) is given by Ahlfors and Grunsky for $\alpha = 1/k$, $k > 0$. But it can be seen that the arguments which they set forth to establish its validity hold also for $\alpha = 0$. The case $\alpha = \frac{1}{3}$ means the mapping of the unit circle on an equilateral straight triangle. We can, therefore, by combining two conformal mappings of the sort (1), map a zero-angled equilateral triangle on an equilateral straight triangle. If we introduce the inverse function

$$\zeta = \Phi_\alpha(w)$$

of $w = \phi_\alpha(\zeta)$ then

$$(2) \qquad w = f(z) = C_1\phi_{1/3}(\Phi_0(C_2 z))$$

will produce that conformal mapping; the constants C_1 and C_2 can be used to adjust the sizes of the triangles.

Now the conformal mapping (2) can be extended by reflection with respect to the sides of the triangles. The repeated images of the zero-angled triangle in the z-plane will just fill its orthogonal circle, which is its circumcircle, whereas the repeated images of the equilateral triangle in the w-plane will build up a Riemann surface in which the vertices of the triangle and of its reflections form a regular point lattice, and where all these vertices become branch-points of infinite order. The function $w = f(z)$ therefore maps, by this process of analytic continuation, the interior of a whole circle K on the described Riemann surface. We can normalize the mapping so that K is the unit circle $|z| < 1$ and so that $f'(0) = 1$.

Since, then, the image of any circle $|z| \leqq 1 - \epsilon$ will not cover any of the branch-points in the Riemann surface over the w-plane a Landau circle cannot be greater than the circumcircle of the equilateral straight triangle which was the element out of which the Riemann surface was constructed.

The normalization just mentioned requires

$$\Phi_0(C_2) = 1$$

or

$$C_2 = \phi_0(1) \tag{3}$$

and

$$f'(0) = C_1^{-1} \frac{\phi'_{1/3}(0)}{\phi'_0(0)} C_2 = 1. \tag{4}$$

Equation (3) has to be understood as

$$C_2 = \lim_{\xi \to 1} \phi_0(\xi) = \lim_{\xi \to 1} \frac{\Gamma(5/6)\Gamma(2/3)F(5/6, 1/2, 4/3; \xi)}{\Gamma(4/3)\Gamma(1/6)F(1/6, 1/2, 2/3; \xi)} .$$

Here we have hypergeometric functions $F(a, b, c; \xi)$ with $a + b = c$, in which case

$$F(a, b, c; \xi) \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log \frac{1}{1-\xi} , \qquad \xi \to 1.$$

Hence

$$C_2 = 1,$$

and therefore formula (4) furnishes

$$C_1 = \frac{\phi'_0(0)}{\phi'_{1/3}(0)} = \frac{\Gamma(5/6)\Gamma(1/3)}{\Gamma(1/6)} .$$

The upper bound of the radii of all Landau circles, i. e. the circumradius of the equilateral triangle in the w-plane, is

$$f(1) = C_1\phi_{1/3}(1) = C_1 = \Gamma(5/6)\Gamma(1/3)\Gamma(1/6)^{-1} = 2\pi\Gamma(1/3)\Gamma(1/6)^{-2}.$$

We have therefore our result:

An upper bound of the Bloch-Landau constant $\mathfrak{L}$ *is given by*

$$\mathfrak{L} \leqq 2\pi\Gamma(1/3)\Gamma(1/6)^{-2}. \tag{5}$$

This result compares numerically with Landau's estimate quoted above as follows:

$$0.54325 < 2\pi\Gamma(1/3)\Gamma(1/6)^{-2} < 0.0.54326,^3$$

$$0.55488 < 2^{2/3}\pi\Gamma(1/3)^3\Gamma(1/4)^{-4} < 0.55489.$$

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[3] In a letter of May 30, 1942, Professor R. M. Robinson informs me that he found the same upper bound for $\mathfrak{L}$ in 1937. His result is briefly mentioned in a footnote to Ahlfor's paper "An extension of Schwarz's Lemma," *Transactions of the American Mathematical Society*, vol. 43 (1938), p. 364 which I had overlooked: "In the other direction R. M. Robinson has proved $\mathfrak{L} < .544$. This result has not been published."

3. The configuration of the zero-angled circular triangle together with all its reflections is, of course, the well-known modular figure; and if it had not been for the convenience of the Ahlfors-Grunsky formula, I would have adhered to my first redaction of the present paper, which was based on the modular function $k^2(\tau) = \lambda(\tau)$ into which $\Phi_0(w)$ can be transformed. Landau in his paper quotes previous results of Carathéodory and Hartogs, which are also obtained from the theory of modular functions. An analysis of Landau's procedure shows that he has actually used the conformal mapping of the zero-angled triangle with the vertices 1, i, -1 on the equilateral triangle. It is through a gain in symmetry of our example over Landau's that we obtained the improved estimate (5).

Ahlfors and Grunsky in their paper make it plausible that their appraisal of Bloch's constant actually gives its true value. A similar reasoning would apply to the above example. The symmetry of the discussed conformal mapping suggests strongly that (5) may contain the best value of the Bloch-Landau constant $\mathfrak{L}$.

It is noteworthy that the Ahlfors-Grunsky example, as well as ours, exhibits functions which have the unit circle as natural boundary. This is in agreement with R. M. Robinson's statements about "Bloch functions,"[4]

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[4] Raphael M. Robinson, "Bloch functions," *Duke Mathematical Journal*, vol. 2 (1936), pp. 453-459.

THE RECTILINEAR MOTION OF A GAS.*

By MONROE H. MARTIN.

Introduction. In the rectilinear motion of a gas the particles of the gas move along straight lines. The lines are all parallel to each other and it is assumed that the velocity and density of the gas at any given instant is constant over any plane perpendicular to the straight lines along which the motion takes place. Letting x denote the distance along any one of these lines measured from some point as origin, and denoting the time by t, the physical problem we consider may be formulated as follows. *Given the velocity u and the density ρ of the gas as a function of x for $t = 0$, what can be said about the subsequent behavior of the gas, i. e., how shall u, ρ be expressed as functions of x, t?* In addition to assuming that the gas moves rectilinearly we shall suppose that it is perfect and that its pressure and density are connected by a characteristic equation $p = p(\rho)$.

This type of motion has been studied by Riemann in a celebrated memoir.[1] In it he reduces the problem of determining the motion of the gas under certain conditions to finding a solution of a linear, hyperbolic partial differential equation of second order of the type $w_{rs} - a(r,s)(w_r - w_s) = 0$, taking prescribed values for its first partial derivatives along a given arc.[2] The solution of this initial value problem given by Riemann has become classic and is known today as "Riemann's method." Essentially the method consists in determining a two parameter family of solutions of the adjoint partial differential equation subject to certain initial conditions. This two-

* Received February 21, 1942; Revised August 22, 1942.

[1] "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite," *Göttingen Abhandlungen*, vol. 8 (1858-59), pp. 245-264 or *Werke*, Leipzig (1892), 2nd Edition, pp. 156-175. See also H. Lamb, *Hydrodynamics*, 5th Ed., Cambridge 1924, pp. 456-459, where both the methods of Riemann and Earnshaw are discussed. For further work on the problem see Rayleigh's *Theory of Sound*, vol. 2, p. 38 and vol. 3, pp. 214-216 of Durand's *Aerodynamic Theory*. An excellent bibliography on the problem is given in the article on Compressible Fluids by H. Bateman in Bulletin 84 of the National Research Council, pp. 543-551. An admirable treatment of the subject will be found in *Die partiellen Differentialgleichungen der Mathematischen Physik* prepared from Riemann's lectures by Heinrich Weber, Braunschweig 1919, vol. 2, pp. 503-562.

An error in § 4 of Riemann's memoir is pointed out in the report of Christoffel, *Fortschritte der Physik*, vol. 15, p. 123. For comments on Lord Rayleigh's objection to Riemann's work see pp. 559-562 of Weber's book cited above.

[2] *Werke*, pp. 160-162 and pp. 169-170.

parameter family of solutions is the so-called "Riemann's function" and, once it has been obtained, the determination of the solution of the original initial value problem is reduced to a quadrature. Since Riemann's time his method has been extended to the most general linear hyperbolic partial differential equation of the second order.[3]

Sections **1** and **2** of the present paper contain no new results but are devoted to a resumé of certain results [4] due to Riemann which the author has felt advisable to include in order to properly prepare the ground for the new developments which follow.

In section **3** the solution of the initial value problem mentioned above is reduced to the determination of what is termed the resolvent of the partial differential equation. The resolvent is analogous to Riemann's function and is a two parameter family of solutions $v = \overline{v}(r, s; \bar{r}, \bar{s})$ of the partial differential equation $v_{rs} + a(r,s)(v_r - v_s) = 0$ satisfying the initial conditions

$$\overline{v}(\bar{r}, s; \bar{r}, \bar{s}) = \bar{s} - s, \qquad \overline{v}(r, \bar{s}; \bar{r}, \bar{s}) = r - \bar{r}.$$

As in Riemann's method, the solution of the initial value problem is reduced to a quadrature, once the resolvent is obtained. The new method offers two advantages. The initial conditions on the resolvent are simpler than those for Riemann's function and the final expression for the solution as a quadrature is simpler in form. On the other hand the method does not appear to permit extension to the general linear, hyperbolic partial differential equation.

The method developed in **3** finds an application in **4** to the solution of the problem of the rectilinear motion of a gas with which Riemann occupied himself in § 10 of his memoir. The culminating point of his work in this paragraph is contained in his equations (2). These equations involve a function ω of four arguments r, s, r', s'. However we do not find any formula in Riemann's paper in which the nature of the dependence of ω upon its four arguments is explicitly set forth.

In the theory developed in the present paper the resolvment $\overline{v}$ plays the role of the function ω of Riemann and in **5**, **6** we calculate $\overline{v}$ explicitly for the important case in which the gas is moving adiabatically. In the determination of $\overline{v}$ we are led to the generalized integral equation of Abel, the solution of which yields a formula for $\overline{v}$ involving Appell's hypergeometric function $F_1(\alpha; \beta, \beta'; \gamma; x, y)$ of two variables.

We have selected in **7** what appears to be mathematically the simplest

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[3] Cf., for example, D. Hilbert und R. Courant, *Methoden der Mathematischen Physik*, II, Berlin (1937), pp. 311-315.

[4] *Werke*, pp. 168-175.

problem to exemplify the application of the methods of this paper to the actual computation of the rectilinear motion of a gas. The application of the method to more complicated cases in which a shock wave develops is reserved for a later paper.

1. The partial differential equations of the problem. If it be assumed that the particles of a gas move in straight lines parallel to the x-axis of a rectangular coördinate system in three dimensional Euclidean space and that the velocity and density of the gas are constant over any plane $x = \text{const.}$ at any given instant t, the partial differential equations for the velocity $u = u(x, t)$ and density $\rho = \rho(x, t)$ are [5]

$$\rho(u_t + u_x u) + G^2 \rho_x = 0, \qquad \rho_t + \rho_x u + \rho u_x = 0, \tag{1}$$

it being understood that there are no external forces. Here

$$G = G(\rho) = (p')^{\frac{1}{2}}, \qquad p = p(\rho), \tag{2}$$

the latter being the characteristic equation of the gas,[6] so that G is the "local velocity of sound." If a new dependent variable f, defined by

$$f = f(\rho) = \int G d\rho/\rho, \tag{3}$$

is introduced in place of ρ, the partial differential equations take the more symmetric form

$$u_t + u_x u + G f_x = 0, \qquad f_t + f_x u + G u_x = 0,$$

which, in turn, may be replaced by

$$r_t + (u + G) r_x = 0, \qquad s_t + (u - G) s_x = 0, \tag{4}$$

on taking

$$u = r + s, \; f = r - s; \qquad 2r = u + f, \; 2s = u - f. \tag{5}$$

Thus u and ρ are functions of $r + s$, $r - s$ respectively. Therefore G, a function of ρ, is likewise a function of $r - s$ and we have

$$u_r = u_s = 1, \qquad G_r = -G_s. \tag{6}$$

Let us consider a solution

$$r = r(x, t), \qquad s = s(x, t), \tag{7}$$

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[5] Literal subscripts denote partial differentiation. Equations (1) follow immediately from the hydrodynamical equations and the equation of continuity on setting

$$X = Y = Z = 0, \qquad v = w = 0, \qquad u_y = u_z = 0, \qquad \rho_y = \rho_z = 0.$$

[6] The prime denotes differentiation with respect to ρ. It is assumed that $p(\rho)$ is a monotone increasing function of ρ possessing a continuous derivative with respect to ρ.

of (4) defined and of class C' throughout a region X of the (x, t)-plane and such that the Jacobian

$$J = \frac{\partial(r, s)}{\partial(x, t)} = 2Gr_x s_x,$$

does not vanish in X. *In addition we suppose the region* X *is mapped in a one-to-one manner upon a region* R *in the* (r, s)*-plane by* (7) and denote by

$$(7)' \qquad x = x(r, s), \qquad t = t(r, s),$$

the inverse transformation from R to X. Now

$$r_t = -Jx_s, \; r_x = Jt_s, \qquad s_t = Jx_r, \; s_x = -Jt_r,$$

and substituting for r_t, r_x, s_t, s_x in (4) we see that the functions (7)′ are solutions of two partial differential equations, linear in the unknown functions x, t

$$(4)' \qquad x_r - (u - G)t_r = 0, \qquad x_s - (u + G)t_s = 0.$$

A solution (7) of (4) enables one to compute the *state of the gas*, that is, the velocity and density of the gas, on a plane $\bar{x}$ at time $\bar{t}$; provided, of course, that the point $(\bar{x}, \bar{t})$ lies in X. On the other hand, the solution (7)′ of (4)′ enables one to compute the location and time of occurrence of a preassigned state; provided, naturally, that the point $(\bar{r}, \bar{s})$ corresponding by (5) to the state in question lies in R. *Either aspect is accordingly available for the description of the motion of the gas.*

Choosing the functions x, t in (7)′ to be solutions of (4)′, the results in (6) may be used to infer that the line integral

$$w = \int \{[x - (u + G)t]dr + [x - (u - G)t]ds\},$$

is independent of the path. Moreover, if $w = w(r, s)$ is the function of r and s thereby defined, we have

$$(7)'' \qquad x - (u + G)t = w_r, \qquad x - (u - G)t = w_s.$$

Subtracting the two equations, and differentiating the first partially with respect to s, we obtain

$$(8) \qquad w_r - w_s = -2Gt, \qquad w_{rs} = (G_r - 1)t,$$

and w *is accordingly a solution of the partial differential equation of the second order*

$$(9) \qquad w_{rs} - \frac{1 - G_r}{2G}(w_r - w_s) = 0.$$

Conversely, if w *is a solution of* (9), *equations* (7)″, *when solved for*

x, t yield functions (7)′ of r, s which are solutions of (4)′. This follows directly upon differentiating the first equation in (7)″ partially with respect to s and the second equation partially with respect to r, the relations (6) being kept in mind. The integration of (4)′ is therefore reduced to the integration of (9).

Differentiating the first equation in (7)″ partially with respect to r, the second partially with respect to s, and using (4)′, one obtains

$$-2Gt_r = w_{rr} + (1 + G_r)t, \qquad 2Gt_s = w_{ss} + (1 + G_r)t.$$

Eliminating t by means of (8), the Jacobian

$$J^{-1} = \frac{\partial(x, t)}{\partial(r, s)} = -2Gt_r t_s,$$

is expressible in terms of w as follows,

$$J^{-1} = \frac{1}{2G}\left[w_{rr} - \frac{1 + G_r}{2G}(w_r - w_s)\right]\left[w_{ss} - \frac{1 + G_r}{2G}(w_r - w_s)\right].$$

By hypothesis the Jacobian J is finite and not equal to zero in the region X. It follows that the Jacobian J^{-1} is not equal to zero and is finite in R. Thus if w denotes the solution of (9) corresponding to the solution (7) of (4), the region R contains none of the loci

$$w_{rr} - \frac{1 + G_r}{2G}(w_r - w_s) = \epsilon, \qquad w_{ss} - \frac{1 + G_r}{2G}(w_r - w_s) = \epsilon, \quad (\epsilon = 0 \text{ or } \infty).$$

2. The initial conditions. If the region X over which the solution (7) of (4) is defined is assumed to contain the segment $\alpha < x < \beta$ of the x-axis, this segment is transformed by (7) into an arc in the (r, s)-plane

$$(10) \qquad C_0: \; r = r_0(x) = r(x, 0), \quad s = s_0(x) = s(x, 0), \quad \alpha < x < \beta,$$

which, since J cannot vanish in X by hypothesis, is met by a line $r = \text{const.}$ (or $s = \text{const.}$) in at most a single point, or, in other words, C_0 is a *monotone* arc.

Thus the initial velocity u_0 and initial density ρ_0 of the gas are assumed to satisfy the inequalities

$$2r'_0 = u'_0 + \frac{G(\rho_0)}{\rho_0}\rho'_0 \neq 0, \quad 2s'_0 = u'_0 - \frac{G(\rho_0)}{\rho_0}\rho'_0 \neq 0, \quad G(\rho_0) \neq 0,$$

along the segment $\alpha < x < \beta$ of the x-axis. Conversely, if the above inequalities are met, the arc C_0 is a monotone arc.

One implication of the assumption that the Jacobian J does not vanish

may now be pointed out. If the gas is initially at rest, the density must be a monotone function of x and the case of wave motion is ruled out.[7] The case in which the Jacobian vanishes identically has been discussed by Riemann[8] and various other writers.

In view of the preceding results, in particular (7)″, the problem of constructing a solution (7) of (4) subject to the initial conditions (10) is equivalent to the following boundary value problem: *to construct a solution w of* (9) *such that the first partial derivatives take values prescribed by*

$$w_r = x, \qquad w_s = x, \tag{11}$$

on the monotone arc C_0.

3. Conjugate partial differential equations. Two partial differential equations of the type

$$L(u) \equiv u_{xy} - a(u_x - u_y) = 0, \quad M(v) \equiv v_{xy} + a(v_x - v_y) = 0, \quad a = a(x, y),$$

are said to be *conjugate to each other*. Taking u, v to be solutions of $L(u) = 0$, $M(v) = 0$ respectively, it is readily verified that

$$(u_x v_x)_y = (u_y v_y)_x,$$

and the integral

$$\int \{u_x v_x dx + u_y v_y dy\},$$

taken around any closed path is accordingly equal to zero.[9] Applying this to the closed path formed by two straight lines $x = \bar{x}$, $y = \bar{y}$ and a monotone arc $\bar{X}\bar{Y}$ connecting a point $\bar{X}$ of $x = \bar{x}$ to a point $\bar{Y}$ of $y = \bar{y}$, one has

$$\int_{\bar{P}}^{\bar{X}} u_y v_y dy + \int_{\bar{X}}^{\bar{Y}} \{u_x v_x dx + u_y v_y dy\} + \int_{\bar{Y}}^{\bar{P}} u_x v_x dx = 0,$$

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[7] In case the density is not a monotone function of x, one may apply the method to the intervals in which the density is monotone. However this will not suffice in general to define the motion of the gas for all x, t. An interesting problem of apparently some difficulty arises when one seeks to prolong the solution over the entire (x, t)-plane.

[8] *Werke*, pp. 162-164. Durand, *Aerodynamic Theory, loc. cit.* See in particular H. Weber, *op. cit.*, pp. 516-522.

[9] Naturally, this statement may require qualification if the closed path bounds a region containing points at which u_x, u_y, v_x, v_y possess singularities. The referee points out that the relation of the method used in the text to classical theory is brought out by the identity

$$(v_x - v_y)L(u) + (u_x - u_y)M(v) = P_y - Q_x, \qquad P = u_x v_x, \qquad Q = u_y v_y,$$

analogous to the well-known identity

$$vL(u) - uM(v) = P_y - Q_x,$$

where M is adjoint to L. Assuming that $L(u) = M(v) = 0$, it is apparent that $\int \{u_x v_x dx + u_y v_y dy\}$ taken around any closed path is zero.

where $\bar{P}$ is the point $(\bar{x}, \bar{y})$. Consequently if a solution $\overline{v}$ of $M(v) = 0$ be taken so that

$$\text{(12)} \qquad \text{on } \bar{P}\bar{X}\colon \overline{v} = \bar{y} - y, \qquad \text{on } \bar{P}\bar{Y}\colon \overline{v} = x - \bar{x},$$

one finds the value $u(\bar{P})$ of the solution u at the point $\bar{P}$ expressed in terms of its values at $\bar{X}$ and $\bar{Y}$, and the values of u_x, u_y along the monotone arc $\bar{X}\bar{Y}$, by the formula

$$\text{(13)} \qquad u(\bar{P}) = \tfrac{1}{2}[u(\bar{X}) + u(\bar{Y})] - \tfrac{1}{2}\int_{\bar{X}}^{\bar{Y}} \{u_x \overline{v}_x dx + u_y \overline{v}_y dy\}.$$

The solution $\overline{v}$ of $M(v) = 0$ employed above obviously depends on the coördinates $\bar{x}$, $\bar{y}$ of the point $\bar{P}$, a fact which we signalize in the notation

$$\overline{v} = \overline{v}(x, y; \bar{x}, \bar{y}) = \overline{v}(P; \bar{P}).$$

The function $\overline{v}$ is termed the *resolvent* of $L(u) = 0$ and the problem of integrating $L(u) = 0$ has been reduced to its determination.

The resolvent $\bar{u}$ of $M(v) = 0$ is a solution of $L(u) = 0$ equal to $y^* - y$ on $x = x^*$ and to $x - x^*$ on $y = y^*$. It is connected with the resolvent $\overline{v}$ of $L(u) = 0$ by the relation

$$\text{(14)} \qquad \bar{u}(\bar{P}; P^*) + \overline{v}(P^*; \bar{P}) = 0,$$

analogous to the Principle of Reciprocity in the theory of the adjoint partial differential equation. To prove this relation, consider the parallelogram with the four vertices

$$\bar{P}(\bar{x}, \bar{y}),\ \bar{X}(\bar{x}, y^*),\ \bar{Y}(x^*, \bar{y}),\ P^*(x^*, y^*).$$

Replacing u by $\bar{u}$ and the arc $\bar{X}\bar{Y}$ in (13) by the broken line $\bar{X}P^*\bar{Y}$, one finds

$$\bar{u}(\bar{P}; P^*) = \tfrac{1}{2}[\bar{u}(\bar{X}; P^*) + \bar{u}(\bar{Y}; P^*)] - \tfrac{1}{2}\int_{\bar{X}}^{P^*} \bar{u}_x(P; P^*)\overline{v}_x(P; \bar{P})dx$$

$$- \tfrac{1}{2}\int_{P^*}^{\bar{Y}} \bar{u}_y(P; P^*)\overline{v}_y(P; \bar{P})dy,$$

and, since by definition

$$\bar{u}_x(P; P^*) = 1 \text{ on } P^*\bar{X}, \qquad \bar{u}_y(P; P^*) = -1 \text{ on } P^*\bar{Y},$$

the relation is established, inasmuch as

$$\bar{u}(\bar{X}; P^*) = \bar{x} - x^*,\ \bar{u}(\bar{Y}; P^*) = y^* - \bar{y},\ \overline{v}(\bar{X}; \bar{P}) = \bar{y} - y^*,\ \overline{v}(\bar{Y}; \bar{P}) = x^* - \bar{x}.$$

Among the relations to be obtained from (14) by partial differentiation with respect to $\bar{x}$, $\bar{y}$, x^*, y^* we shall set down only two

$$(15)\qquad \bar{u}_{\bar{x}}(\bar{P};P^*)+\bar{v}_{\bar{x}}(P^*;\bar{P})=0,\qquad \bar{u}_{\bar{y}}(\bar{P};P^*)+\bar{v}_{\bar{y}}(P^*;\bar{P})=0,$$

as important for later application.

Returning to the solution (13) and differentiating it partially with respect to $\bar{x}$ and $\bar{y}$, it is found that

$$(16)\qquad u_{\bar{x}}=\tfrac{1}{2}u_x(\bar{X})[1+\bar{v}_x(\bar{X})]-\tfrac{1}{2}\int_{\bar{X}}^{\bar{Y}}\{u_x\bar{v}_{x\bar{x}}dx+u_y\bar{v}_{y\bar{x}}dy\},$$

$$(17)\qquad u_{\bar{y}}=\tfrac{1}{2}u_y(\bar{Y})[1-\bar{v}_y(\bar{Y})]-\tfrac{1}{2}\int_{\bar{X}}^{\bar{Y}}\{u_x\bar{v}_{x\bar{y}}dx+u_y\bar{v}_{y\bar{y}}dy\}.$$

It will be recalled that (14) was derived from (13) by replacing u by $\bar{u}$ and the arc $\bar{X}\bar{Y}$ by the broken line $\bar{X}P^*\bar{Y}$. If the same procedure is applied to (16) and (17), it is found that

$$\bar{v}_x(\bar{X};\bar{P})+\bar{v}_{\bar{x}}(\bar{X};\bar{P})=2[\bar{u}_{\bar{x}}(\bar{P};P^*)+\bar{v}_{\bar{x}}(P^*;\bar{P})],$$
$$\bar{v}_y(\bar{Y};\bar{P})+\bar{v}_{\bar{y}}(\bar{Y};\bar{P})=2[\bar{u}_{\bar{y}}(\bar{P};P^*)+\bar{v}_{\bar{y}}(P^*;\bar{P})],$$

inasmuch as

$$(18)\qquad \bar{v}_{\bar{x}}(\bar{Y};\bar{P})=-1,\qquad \bar{v}_{\bar{y}}(\bar{X};\bar{P})=1.$$

Referring to (15) it is apparent that

$$\bar{v}_x(\bar{X};\bar{P})+\bar{v}_{\bar{x}}(\bar{X};\bar{P})=0,\qquad \bar{v}_y(\bar{Y};\bar{P})+\bar{v}_{\bar{y}}(\bar{Y};\bar{P})=0,$$

which, since $\bar{X}$ may be taken arbitrarily on $x=\bar{x}$ and $\bar{Y}$ may be taken arbitrarily on $y=\bar{y}$, leads to the following principle. *The resolvent* $\bar{v}=\bar{v}(x,y;\bar{x},\bar{y})$ *satisfies the relations*

$$(19)\qquad \bar{v}_x+\bar{v}_{\bar{x}}=0,\qquad \bar{v}_y+\bar{v}_{\bar{y}}=0,$$

respectively along the lines $x=\bar{x}$ *and* $y=\bar{y}$.

4. Rectilinear motion of a gas. In the first two sections of this paper the problem of determining the rectilinear motion of a gas under the hypothesis stated therein has been reduced to the boundary value problem formulated at the close of **2**. For any equation of state $p=p(\rho)$ the partial differential equation (9) takes the form

$$(20)\qquad w_{rs}-a(w_r-w_s)=0,\qquad a=a(r,s),$$

and, granting that the resolvent,

$$\bar{v}=\bar{v}(r,s;\bar{r},\bar{s}),$$

of this partial differential equation has been determined, the value of the solution w at a point $(\bar{r}, \bar{s})$ is, from (13), computed by

$$w(\bar{r}, \bar{s}) = \tfrac{1}{2}[w(\bar{R}) + w(\bar{S})] - \tfrac{1}{2}\int_{\bar{R}}^{\bar{S}} \{w_r \overline{v}_r dr + w_s \overline{v}_s ds\},$$

in which $\bar{R}$, $\bar{S}$ denote the points where [10] the lines $r = \bar{r}$, $s = \bar{s}$, respectively, intersect the monotone arc C_0.

What is really needed, however, in (7″), is not the value of the function w at the point $(\bar{r}, \bar{s})$, but the values of its partial derivatives w_r, w_s at this point. For these we refer to (16), (17) and obtain

$$w_{\bar{r}} = \tfrac{1}{2} w_r(\bar{R})[1 + \overline{v}_r(\bar{R})] - \tfrac{1}{2}\int_{\bar{R}}^{\bar{S}} \{w_r \overline{v}_{r\bar{r}} dr + w_s \overline{v}_{s\bar{r}} ds\},$$

$$w_{\bar{s}} = \tfrac{1}{2} w_s(\bar{S})[1 - \overline{v}_s(\bar{S})] - \tfrac{1}{2}\int_{\bar{R}}^{\bar{S}} \{w_r \overline{v}_{r\bar{s}} dr + w_s \overline{v}_{s\bar{s}} ds\},$$

which, when the initial values (11) for w_r, w_s on C_0 are inserted, become

$$w_{\bar{r}} = \tfrac{1}{2} x(\bar{R})[1 + \overline{v}_r(\bar{R})] - \tfrac{1}{2}\int_{\bar{R}}^{\bar{S}} x d\overline{v}_r, \quad w_{\bar{s}} = \tfrac{1}{2} x(\bar{S})[1 - \overline{v}_s(\bar{S})] - \tfrac{1}{2}\int_{\bar{R}}^{\bar{S}} x d\overline{v}_s,$$

where $x(\bar{R})$, $x(\bar{S})$ denote the abscissae of the points on the interval $\alpha < x < \beta$ of the x-axis which map into the points $\bar{R}$, $\bar{S}$ under the transformation (10). After integrating by parts and applying (18) and (19) it is found that

$$w_r = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \overline{v}_r dx, \quad w_{\bar{s}} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \overline{v}_s dx.$$

The motion of the gas is consequently to be determined from the following equations

$$\bar{x} - (\bar{u} + \bar{G})\bar{t} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \overline{v}_r dx, \tag{21}$$

$$\bar{x} - (\bar{u} - \bar{G})\bar{t} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \overline{v}_s dx,$$

obtained by substituting for the partial derivatives in (7)″.

[10] The values of w at the points $\bar{R}$, $\bar{S}$ are indicated by $w(\bar{R})$, $w(\bar{S})$ respectively.

The use of these formulae may best be understood with the aid of Figure 1. Suppose the initial velocity and density u_0, ρ_0 of the gas are known along the segment $\alpha < x < \beta$ of the x-axis. To fix the ideas, let us suppose that along this segment u_0, ρ_0 are chosen so that the arc C_0 of **2** is the monotone arc $A\bar{R}\bar{S}B$ of the figure. The basic idea in the use of these formulae is that they yield a mapping of the "triangular region" ABC of the (r, s)-plane upon the "triangular region" $\alpha\beta\gamma$ of the (x, t)-plane. Thus the mapping brings to each point $(\bar{x}, \bar{t})$ inside $\alpha\beta\gamma$ a pair of values $\bar{r}, \bar{s}$ from which the velocity $\bar{u}$ and density $\bar{\rho}$ of the gas at the point $(\bar{x}, \bar{t})$ is computed from (5) and (3).

The details of the application of these formulae will be taken up in **7** where they are applied to a specific problem in the rectilinear motion of a gas.

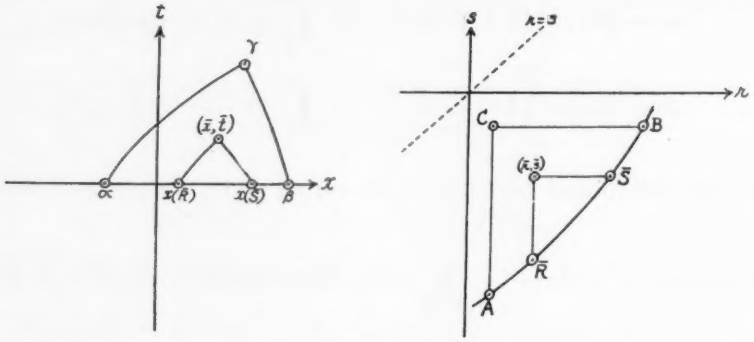

Figure 1.

5. The adiabatic case. The results obtained so far are valid independent of the choice made for the function $p(\rho)$ expressing the dependence of the pressure of a gas upon its density. From now on we shall suppose that the motion of the gas proceeds adiabatically. The characteristic equation of the gas takes the well-known form

$$p = k^2\rho^\gamma,$$

where γ is the ratio of the two specific heats of the gas and k is a constant. It is readily verified that

$$G = \frac{\gamma - 1}{2}(r - s),$$

and that the partial differential equation (9) becomes

$$(22) \qquad (r - s)w_{rs} - m(w_r - w_s) = 0, \qquad m = \frac{3 - \gamma}{2(\gamma - 1)}.$$

This type of partial differential equation has received intensive study [11] since the time of Euler.

Equations (21) controlling the motion of the gas take the form

$$(23)\qquad \bar{x} - \left(\frac{1+\gamma}{2}\bar{r} + \frac{3-\gamma}{2}\bar{s}\right)\bar{t} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \overline{v_r} dx,$$

$$\bar{x} - \left(\frac{3-\gamma}{2}\bar{r} + \frac{1+\gamma}{2}\bar{s}\right)\bar{t} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \overline{v_s} dx,$$

and, once the resolvent $\overline{v}$ of (22) has been found, may be used to compute the motion of the gas as explained in **4**.

According to the kinetic theory of gases [12]

$$\gamma = 1 + \frac{2}{3+n},$$

where $n = 0, 2$ for monatomic and diatomic gases respectively, a result which is in good agreement with experiment. The corresponding value of m is

$$m = \tfrac{1}{2}(n+2),$$

so that $m = 1$ for monatomic gases and $m = 2$ for diatomic gases. There appears [12] to be no known law for n in the case of more complex gases, and we are accordingly led to consider the problem of integrating (22) for arbitrary constant values of m.

The general solutions of (22) for monatomic and diatomic gases are

$$w = \frac{R-S}{r-s}, \qquad w = \frac{\partial^2}{\partial r \partial s}\,\frac{R-S}{r-s},$$

where R denotes an arbitrary function of r and S an arbitrary function of s. Although the general solution of (22) is available [11] for arbitrary constant values of m, our problem is to find a solution subject to the initial conditions (11). Presumably one might achieve this by properly specializing the arbitrary elements in the general solution. However it appears more feasible to proceed along the lines of **4** and we accordingly take up the problem of determining the resolvent of (22).

6. The resolvent. The resolvent $\overline{v}$ of (22) is a two parameter family $\overline{v} = \overline{v}(r, s; \bar{r}, \bar{s})$ of solutions of the conjugate equation

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[11] See, for example, G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, 2nd Ed., Paris (1915), pp. 54-70.

[12] J. H. Jeans, *The Dynamical Theory of Gases*, 4th Ed., Cambridge (1925), pp. 191-193. Recent work by D. H. Andrews has enabled fairly accurate estimates of γ to be made for molecules containing up to 20 atoms.

$$(24) \qquad (r-s)v_{rs} + m(v_r - v_s) = 0,$$

such that

$$(25) \qquad \overline{v}(\bar{r}, s; \bar{r}, \bar{s}) = \bar{s} - s, \qquad \overline{v}(r, \bar{s}; \bar{r}, \bar{s}) = r - \bar{r}.$$

It is easily verified [13] that a one parameter family of solutions of (24) is given by

$$v = (r-a)^m(a-s)^m,$$

a being the parameter of the family. This one parameter family offers the starting point for our construction of the resolvent.

To begin with we observe in (3) that f is a positive function of ρ and therefore it follows from (5) that $r > s$. We may accordingly confine our attention to the half plane lying under the line $r = s$. Taking $\bar{r} > \bar{s}$, we shall construct the resolvent of (22) under the following restrictions

$$(26) \qquad r > \bar{r} > 0,\ s < \bar{s}; \qquad m = -\lambda \qquad (0 < \lambda < 1).$$

Eventually we shall show how these restrictions may be removed.

By forming the solution [14]

$$v = \int_s^r \phi(a)\,(r-a)^{-\lambda}(a-s)^{-\lambda}da,$$

where $\phi(a)$ is an arbitrary function we gain a solution of (24) containing an arbitrary function. Taking $\phi(a) = 0$ for $\bar{s} \leqq a \leqq \bar{r}$, we arrive at a two-parameter family of solutions

$$(27) \quad v = \int_{\bar{r}}^r \phi(a)\,(r-a)^{-\lambda}(a-s)^{-\lambda}da + \int_s^{\bar{s}} \phi(a)\,(r-a)^{-\lambda}(a-s)^{-\lambda}da,$$

involving an arbitrary function. If this arbitrary function can now be determined so that the conditions laid down in (25) are met, the resolvent of (22) will have been determined.

Thus we are to determine $\phi(a)$ so that

$$(28) \quad \int_{\bar{r}}^r \phi(a)\,(a-\bar{s})^{-\lambda}(r-a)^{-\lambda}da = r - \bar{r}, \quad \int_{\bar{s}}^s \phi(a)\,(\bar{r}-a)^{-\lambda}(a-s)^{-\lambda}da = s -$$

Let us consider the first of these integral equations. If we define

$$\psi(t) = \Psi(t) = 0,\ 0 \leqq t \leqq \bar{r}; \quad \psi(t) = \phi(t)\,(t-\bar{s})^{-\lambda},\ \Psi(t) = t - \bar{r},\ t \geqq \bar{r},$$

this integral equation is recognized as the generalized integral equation of Abel

$$\int_0^r \psi(t)\,(r-t)^{-\lambda}dt = \Psi(t).$$

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[13] Cf. G. Darboux, *op. cit.*, p. 58.

[14] Cf. G. Darboux, *op. cit.*, pp. 66-68.

The solution of this integral equation is [15]

$$\psi(a) = \frac{\sin \pi\lambda}{\pi} \int_0^a \Psi'(t)\,(a-t)^{\lambda-1}dt = \frac{\sin \pi\lambda}{\pi} \int_{\bar{r}}^a (a-t)^{\lambda-1}dt,$$

from which it is readily ascertained that

$$\phi(a) = \frac{\sin \pi\lambda}{\pi\lambda}\,(a-\bar{r})^{\lambda}(a-\bar{s})^{\lambda}. \tag{29}$$

The second integral equation goes over into the first when the variables in the pairs (r, s), $(\bar{r}, \bar{s})$ are interchanged. It follows that the solution is also given by (29).

When $\phi(a)$ is substituted in (27) and a new integration variable t is introduced by setting

$$a = \bar{r} + t(r-\bar{r}), \qquad a = \bar{s} + t(s-\bar{s}),$$

in the first and second integrals respectively, the resolvent $\overline{v}$ is obtained in the form

$$\overline{v} = \frac{\sin \pi\lambda}{\pi\lambda} \left\{ (r-\bar{r}) \int_0^1 \frac{t^{\lambda}[\bar{r}-\bar{s}+t(r-\bar{r})]^{\lambda}}{(1-t)^{\lambda}[\bar{r}-s+t(r-\bar{r})]^{\lambda}}\,dt \right.$$
$$\left. + (\bar{s}-s) \int_0^1 \frac{t^{\lambda}[\bar{r}-\bar{s}-t(s-\bar{s})]^{\lambda}}{(1-t)^{\lambda}[r-\bar{s}-t(s-\bar{s})]^{\lambda}}\,dt \right\}.$$

The resolvent will now be expressed in terms of Appell's hypergeometric function of two variables [16]

$$F_1(\alpha;\, \beta, \beta';\, \gamma;\, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)\,(\beta, m)\,(\beta', n)}{(\gamma, m+n)\,(1, m)\,(1, n)}\, x^m y^n, \tag{30}$$
$$(\alpha, r) = \alpha(\alpha+1)\cdots(\alpha+r-1).$$

Indeed if $\Re(\alpha) > 0$, $\Re(\gamma-\alpha) > 0$, Picard has shown [17] that

$$\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}dt = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\, F_1(\alpha;\beta,\beta';\gamma;x,y).$$

If we put $\alpha = 1+\lambda$, $\beta = -\lambda$, $\beta' = \lambda$, $\gamma = 2$, in this formula and observe that

$$\frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(2)} = \frac{\pi\lambda}{\sin \pi\lambda},$$

we find on writing

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[15] The solution of this integral equation is well known. Cf., for example, G. Kowalewski, *Integralgleichungen*, Leipzig (1930), pp. 11-13.

[16] P. Appell, "Sur les fonctions hypergéométriques de plusieurs variables," *Memorial des Sciences Mathématiques*, Fasc. III (1925), p. 4.

[17] See Appell, *op. cit.*, p. 9.

$$x = \frac{r-\bar{r}}{\bar{s}-\bar{r}}, \quad y = \frac{r-\bar{r}}{s-\bar{r}}; \quad \text{and} \quad x = \frac{s-\bar{s}}{\bar{r}-\bar{s}}, \quad y = \frac{s-\bar{s}}{r-\bar{s}},$$

that the resolvent may be expressed as follows

$$(31) \quad \overline{v} = (r-\bar{r})\left(\frac{\bar{r}-\bar{s}}{\bar{r}-s}\right)^{\lambda} F_1\left(1+\lambda; -\lambda, \lambda; 2; \frac{r-\bar{r}}{\bar{s}-\bar{r}}, \frac{r-\bar{r}}{s-\bar{r}}\right)$$
$$-(s-\bar{s})\left(\frac{\bar{r}-\bar{s}}{r-\bar{s}}\right)^{\lambda} F_1\left(1+\lambda; -\lambda, \lambda; 2; \frac{s-\bar{s}}{\bar{r}-\bar{s}}, \frac{s-\bar{s}}{r-\bar{s}}\right).$$

It will be recalled that the resolvent was derived under the restrictions (26). These restrictions we now proceed to remove. Availing ourselves of the formula [18]

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha; \beta, \beta'; \gamma; \frac{x}{x-1}, \frac{y}{y-1}\right),$$

we find alternative expressions for the hypergeometric functions in (31), namely,

$$F_1\left(1+\lambda; -\lambda, \lambda; 2; \frac{r-\bar{r}}{\bar{s}-\bar{r}}, \frac{r-\bar{r}}{s-\bar{r}}\right)$$
$$= \left(\frac{r-\bar{s}}{\bar{r}-\bar{s}}\right)^{\lambda}\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda} F_1\left(1-\lambda; -\lambda, \lambda; 2; \frac{r-\bar{r}}{r-\bar{s}}, \frac{r-\bar{r}}{r-s}\right),$$

$$F_1\left(1+\lambda; -\lambda, \lambda; 2; \frac{s-\bar{s}}{\bar{r}-\bar{s}}, \frac{s-\bar{s}}{r-\bar{s}}\right)$$
$$= \left(\frac{\bar{r}-s}{\bar{r}-\bar{s}}\right)^{\lambda}\left(\frac{r-\bar{s}}{r-s}\right)^{\lambda} F_1\left(1-\lambda; -\lambda, \lambda; 2; \frac{s-\bar{s}}{s-\bar{r}}, \frac{s-\bar{s}}{s-r}\right),$$

and thus we are led to consider three additional forms for $\overline{v}$:

$$(32) \quad \overline{v} = (r-\bar{r})\left(\frac{r-\bar{s}}{r-s}\right)^{\lambda} F_1\left(1-\lambda; -\lambda, \lambda, 2; \frac{r-\bar{r}}{r-\bar{s}}, \frac{r-\bar{r}}{r-s}\right)$$
$$-(s-\bar{s})\left(\frac{\bar{r}-\bar{s}}{r-\bar{s}}\right)^{\lambda} F_1\left(1+\lambda; -\lambda, \lambda; 2; \frac{s-\bar{s}}{\bar{r}-\bar{s}}, \frac{s-\bar{s}}{r-\bar{s}}\right),$$

$$(33) \quad \overline{v} = (r-\bar{r})\left(\frac{\bar{r}-\bar{s}}{\bar{r}-s}\right)^{\lambda} F_1\left(1+\lambda; -\lambda, \lambda; 2; \frac{r-\bar{r}}{\bar{s}-\bar{r}}, \frac{r-\bar{r}}{s-\bar{r}}\right)$$
$$-(s-\bar{s})\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda} F_1\left(1-\lambda; -\lambda, \lambda, 2; \frac{s-\bar{s}}{s-\bar{r}}, \frac{s-\bar{s}}{s-r}\right),$$

$$(34) \quad \overline{v} = (r-\bar{r})\left(\frac{r-\bar{s}}{r-s}\right)^{\lambda} F_1\left(1-\lambda; -\lambda, \lambda, 2; \frac{r-\bar{r}}{r-\bar{s}}, \frac{r-\bar{r}}{r-s}\right),$$
$$-(s-\bar{s})\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda} F_1\left(1-\lambda; -\lambda, \lambda, 2; \frac{s-\bar{s}}{s-\bar{r}}, \frac{s-\bar{s}}{s-r}\right).$$

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[18] See, for example, W. N. Bailey, "Generalized hypergeometric series," *Cambridge Tracts in Mathematics and Physics*, No. 32, Cambridge (1935), p. 78.

Appell [19] has shown that the power series (30) converges for *arbitrary real or complex values of* α, β, γ inside the unit square $|x| < 1$, $|y| < 1$, provided, of course, γ is not a negative integer. The various formulae obtained for $\overline{v}$ accordingly define the resolvent in the following regions respectively:

$$(31)' \quad |r-\bar{r}| < |\bar{r}-\bar{s}|, |r-\bar{r}| < |s-\bar{r}|, |s-\bar{s}| < |\bar{r}-\bar{s}|, |s-\bar{s}| < |r-\bar{s}|;$$

$$(32)' \quad |r-\bar{r}| < |r-\bar{s}|, |r-\bar{r}| < |r-s|, |s-\bar{s}| < |\bar{r}-\bar{s}|, |s-\bar{s}| < |r-\bar{s}|;$$

$$(33)' \quad |r-\bar{r}| < |\bar{r}-\bar{s}|, |r-\bar{r}| < |s-\bar{r}|, |s-\bar{s}| < |s-\bar{r}|, |s-\bar{s}| < |s-r|;$$

$$(34)' \quad |r-\bar{r}| < |r-\bar{s}|, |r-\bar{r}| < |r-s|, |s-\bar{s}| < |s-\bar{r}|, |s-\bar{s}| < |s-r|.$$

The four forms (31), (32), (33), (34) obtained for $\overline{v}$ are analytic continuations of each other and taken together suffice to define the resolvent throughout the region

$$s < \bar{r}, \quad r > s, \quad r > \bar{s},$$

for any real value of λ. By hypothesis the arc $\bar{R}\bar{S}$ in (21) or (23) connecting a point of $r = \bar{r}$ to a point of $s = \bar{s}$ is a monotone arc lying in the region $r > s$ and therefore cannot leave the above region. It follows that $\overline{v}$ is adequately determined for use in (21) or (23).

The formula [20]

$$F_1(\alpha; \beta, \beta'; \gamma; x, x) = F(\alpha, \beta + \beta', \gamma; x),$$

reducing the hypergeometric function F_1 to an ordinary hypergeometric function affords a partial check on our computation, inasmuch as, taking $r = \bar{r}$ and $s = \bar{s}$ in (31) yields

$$\overline{v}(\bar{r}, s; \bar{r}, \bar{s}) = (\bar{s} - s) F\left(1 + \lambda, 0, 2; \frac{s-\bar{s}}{\bar{r}-\bar{s}}\right) = \bar{s} - s,$$

$$\overline{v}(r, \bar{s}; \bar{r}, \bar{s}) = (r - \bar{r}) F\left(1 + \lambda, 0, 2; \frac{r-\bar{r}}{\bar{s}-\bar{r}}\right) = r - \bar{r},$$

which agrees with (25).

A further check is provided by the monatomic gases, the partial differential equation (22) becoming

$$(35) \qquad (r-s) w_{rs} - (w_r - w_s) = 0.$$

Taking $\lambda = -1$ in (31), the resolvent of (35) is calculated with the aid of (30) and found to be

$$(36) \qquad \overline{v} = \frac{(r-\bar{r})(\bar{r}-s) - (s-\bar{s})(r-\bar{s})}{\bar{r}-\bar{s}},$$

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[19] *Op. cit.*, p. 4.

[20] P. Appell, *op. cit.*, pp. 3-4.

and a direct computation verifies that this function $\bar{v}$ is a solution of the conjugate equation to (35) and that $\bar{v}$ fulfills the conditions required in (25).

7. The motion of a monatomic gas. Of all the gases the motion of a monatomic gas appears to be most susceptible to mathematical investigation. Thus the resolvent $\bar{v}$ has the particularly simple form (36). One readily verifies that placing this resolvent in (23) with $\gamma = 5/3$ leads to the following explicit formulae:

$$\bar{x} = (\bar{r} - \bar{s})^{-3}\{\bar{r}(\bar{r}^2 + \bar{r}\bar{s} + 4\bar{s}^2)x(\bar{R}) - \bar{s}(4\bar{r}^2 + \bar{r}\bar{s} + \bar{s}^2)x(\bar{S}) - \int_{x(\bar{R})}^{x(\bar{S})} [3(\bar{r} + \bar{s})rs - 2(\bar{r}^2 + \bar{r}\bar{s} + \bar{s}^2)(r + s)]dx\},$$

$$\bar{t} = 3(\bar{r} - \bar{s})^{-3}\{\bar{r}\bar{s}[x(\bar{R}) - x(\bar{S})] - \int_{x(\bar{R})}^{x(\bar{S})} [rs - \tfrac{1}{2}(\bar{r} + \bar{s})(r + s)]dx\}.$$

To illustrate the application of the method to a concrete problem let us investigate the rectilinear motion of a monatomic gas which initially is at rest everywhere, a vacuum ($\rho = 0$) being present for negative x and $\rho = (2x/15k^2)^{3/2}$ for positive x, under the assumption that the gas moves adiabatically. The apparent artificiality in the choice of the initial density due to an attempt on our part to find an example which illustrates the method without burdening the reader with tedious and excessive computation.

Referring to **1**, one verifies that $f = \sqrt{15}\, k\rho^{1/3}$ and that the parametric equations (10) of the arc C_0 in **2** (the arc $A\bar{R}\bar{S}B$ in Figure 1) are

$$r = (x/2)^{1/2}, \qquad s = -(x/2)^{1/2}.$$

Substituting $x(\bar{R}) = 2\bar{r}^2$, $x(\bar{S}) = 2\bar{s}^2$ and carrying out the integration in the above formulae, we find on dropping the bars from all letters that the region $r \geqq s$ of the (r, s)-plane is mapped upon a region in the (x, t)-plane by

$$x = -(r^2 + 4rs + s^2), \qquad t = -3(r + s).$$

These formulas take on greater physical significance if the variables u, f are employed in place of r, s. In terms of these variables, the mapping is given by

$$x = \tfrac{1}{2}(f^2 - 3u^2), \qquad t = -3u. \tag{37}$$

It is apparent that the loci in the (x, t)-plane of points where the gas has a constant velocity comprise the lines $t = \text{const.}$, while the loci of points where the gas has constant density are the parabolas $x = \frac{1}{2}(f^2 - t^2/3)$, $f = \text{const.}$ These curves are graphed below and form a network over a region in the (x, t)-plane, from which it is possible to read off the velocity and density of

the gas at any position and at any time. Outside this region both the density and velocity of the gas is zero, i. e., physically one has a vacuum. As a check on our computation, one finds on solving (37) for u, f in terms of x, t, that

$$u = -t/3, \qquad \rho = \left(\frac{2x + t^2/3}{15k^2}\right)^{3/2},$$

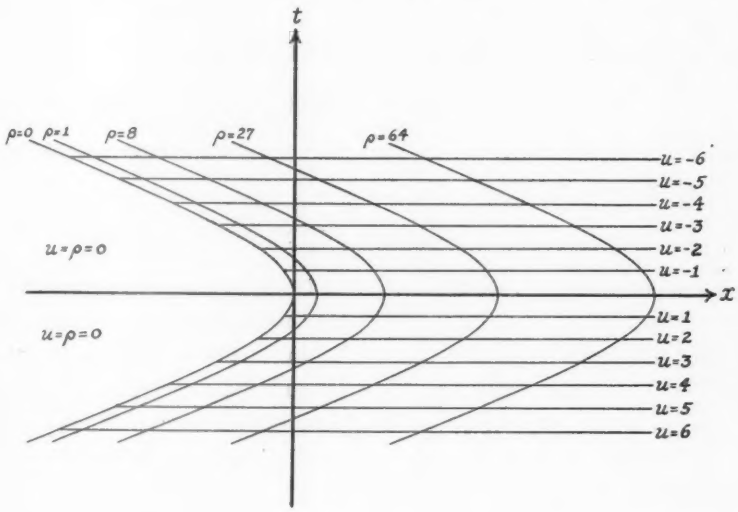

Figure 2.

yield a solution of the hydrodynamical equations (1). It will be noted that a discontinuity in the velocity presents itself along the parabola $\rho = 0$ of Figure 2. The discontinuity meets, however, the compatibility, impulse, and energy conditions.[21]

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[21] For a formulation of these conditions, see Weber, *op. cit.*, p. 511.

A CLASS OF SOLUTIONS FOR THE HEAT EQUATION AND ASSOCIATED BOUNDARY VALUE PROBLEMS.*

By F. G. DRESSEL and E. R. ELLIOTT.

1. Introduction. Associated with the heat equation

$$\partial^2 u/\partial x^2 = \partial u/\partial y, \tag{1}$$

we have the region D bounded by the characteristics whose ordinates are h and g $(h < g)$, and the curves

$$x = \chi_1(y), \qquad x = \chi_2(y), \qquad h \leq y \leq g,\ \chi_1 < \chi_2.$$

The problem of finding a solution of (1) in D, taking on preassigned continuous values on the boundary of D [1] has been treated by several authors. Holmgren [2] solved the problem, using integral equations, under the assumption that the $\chi_i(\eta)$ had continuous derivatives. Later papers have sought to weaken the hypothesis on the $\chi_i(\eta)$ [3] or to use other methods for the solution.[4]

In the first part of this paper we treat a problem similar to that of a former paper,[5] where certain line integrals of solutions of (1) were made to take on boundary values of limited variation.[6] In Part 2 of the present paper,

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[1] No boundary values are assigned on the line $y = g$.

[2] E. Holmgren, "Sur une application de l'équation intégral de M. Volterra," *Arkiv för Matematik*, vol. 3 (1906-07), pp. 1-4.

[3] M. Gevrey, "Sur les équations aux dériveés partielles du type parabolique," *Journal de Mathématiques*, (6), vol. 9 (1913), pp. 305-471. Gevrey showed that, if the $\chi_i(\eta)$ satisfy a Lipschitz condition of order $\alpha > \frac{1}{2}$ the problem has a solution.

[4] W. Sternberg, "Über die Gleichung der Wärmeleitung," *Mathematische Annalen*, vol. 101 (1929), pp. 394-398. Here the problem was solved by Perron's method under the assumption that the derivatives of the $\chi_i(\eta)$ were bounded. I. Petrowsky ["Zur ersten Randwertaufgabe der Wärmeleitunggleichung," *Compositio Mathematica*, vol. 1 (1934-35), pp. 383-419] improved Sternberg's treatment and showed that a weaker condition than that of Gevrey was sufficient.

[5] F. G. Dressel, "A boundary value problem for the heat equation," *American Journal of Mathematics*, vol. 55 (1933), pp. 641-653. This will be referred to as paper (A).

[6] The results of Part 1 of this paper are taken from E. R. Elliott's doctoral dissertation presented while in the graduate school of Duke University. The investigation was suggested by Professor E. R. C. Miles, to whom Mr. Elliott is indebted for suggestions and encouragement.

we show how to construct a solution of equation (1) in D which takes on preassigned functions of limited variation on the boundary. However, the approaches to the boundary of D are in a restricted sense that we call parabolic approaches.

PART 1

In this part of the paper we shall show how to construct a solution $u(x, y)$ in the region D satisfying boundary conditions of the following type:

$$\lim_{\delta \to 0} \int_{x_1}^{x_2} u(t, \delta)\, dt = B(x_2) - B(x_1), \qquad \delta > 0, \tag{2}$$

$$\lim_{\lambda_i \to 0} \int_{y_1}^{y_2} u_x(\chi_i(t) + \lambda_i, t)\, dt = G_i(y_2) - G_i(y_1), \quad (-1)^{i+1}\lambda_i > 0, \quad i = 1, 2, \tag{3}$$

where $B(x)$ and $G_i(y)$ are preassigned functions of limited variation. The problem differs from that in paper (A) in that here the line integral of u_x parallel to the curves χ_i is used in place of the line integral of the solution u itself. To carry through some of the proofs we have found it necessary to replace the Riemann-Stieltjes integral used in our former paper by the Young-Stieltjes integral.

Before stating our principal result we shall prove several introductory theorems about the class of functions we are using.

2. Lemmas. We first prove a lemma on the change of order of integration.

LEMMA 1. *Let $G(x, y)$ be continuous in $m \leq x \leq n$, $p \leq y \leq q$, with the possible exception of the point (a, b), $m < a < n$, $p < b < q$. Let*

$$\left| \int_m^n G(x, y)\, dx \right| < M, \qquad p \leq y \leq q.$$

If $F(y)$ is of limited variation, then

$$\int_m^n dx \cdot \int_p^q G(x, y)\, dF(y) = \int_p^q dF(y) \cdot \int_m^n G(x, y)\, dx,$$

when the above integrals are considered in the Young-Stieltjes sense.

From the hypothesis, it follows that $\int G(x, y)\, dx$ exists for each y, is bounded in y and continuous except for $y = b$; hence

$$\int_p^q dF(y) \cdot \int_m^n G(x, y)\, dx \tag{4}$$

exists in the Young-Stieltjes sense. Further by noting that $G(x, y)$ is continuous for $m \leq x \leq a - \delta$, $p \leq y \leq q$, $(\delta > 0)$, we have

$$\int_m^{a-\delta} dx \cdot \int_p^q G(x,y)\,dF(y) = \int_p^q dF(y) \cdot \int_m^{a-\delta} G(x,y)\,dx.$$

As $\delta \to 0$, the integral on the right is seen to have a bounded limit. A similar treatment of the interval $a \leq x \leq n$ shows that the integral

$$\int_m^n dx \cdot \int_p^q G(x,y)\,dF(y) \tag{5}$$

exists. Since the two integrals (4) and (5) exist, they are equal by a theorem of W. H. Young.[7]

The fundamental solution of (1) will be denoted by

$$U(x,y) = [4\pi y]^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4y}\right),\ y > 0; \qquad = 0,\ y \leq 0.$$

By $V(x,y)$ we shall mean the function

$$V(x,y) = -2U_x(x,y),\ y > 0; \qquad = 0,\ y \leq 0, \tag{6}$$

which is a solution of (1) for $y > 0$.

We shall now prove several theorems about solutions of the heat equation constructed from the functions U and V.

THEOREM 1. *Let $F(y)$ be of limited variation on $h \leq y \leq g$, and continuous at h. Let $\chi(y)$ be continuous on the same interval. Then*

$$I(\delta) \equiv \int_{x_1}^{x_2} dx \cdot \int_h^{h+\delta} U(x - \chi(\eta),\ h + \delta - \eta)\,df(\eta) \to 0, \text{ with } \delta \to 0.$$

$I(\delta)$ is the line integral of a solution of (1) along the line $y = h + \delta$. Applying our first lemma, we have

$$|I(\delta)| = \left| \int_h^{h+\delta} dF(\eta) \cdot \int_{x_1}^{x_2} U(x - \chi(\eta),\ h + \delta - \eta)\,dx \right|$$
$$\leq M \cdot T(h, h+\delta),$$

where M is an upper bound for the integral of U with respect to x and $T(h, h+\delta)$ denotes the total variation of $F(y)$ over $h \leq y \leq y + \delta$. $T(h, h+\delta)$ approaches zero with δ, since $F(y)$ is continuous at $y = h$; hence

$$\lim_{\delta \to 0} I(\delta) = 0.$$

For completeness we state

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[7] W. H. Young, "Integration with respect to a function of bounded variation," *Proceedings of the London Mathematical Society*, Series 2, vol. 13 (1914), pp. 109-150. See p. 148.

THEOREM 2. *Let $B(x)$ be of limited variation on $a \leq x \leq b$, continuous at a and b, and have regular discontinuities for $a < x < b$. Then*

$$\lim_{\delta \to 0} \int_{x_1}^{x_2} dx \cdot \int_a^b U(x - \xi, \delta) dB(\xi) = B(x_2) - B(x_1).$$

This follows from Theorem 2 of paper (A).

THEOREM 3. *Let $\chi'(y)$ be continuous on $h \leq y \leq g$, and let $B(x)$ be of limited variation on $a \leq x \leq b$ and continuous at $x = \chi(h)$ $(a \leq \chi(h) \leq b)$. We then have*

$$\lim_{\lambda \to 0} \int_h^y dt \cdot \int_a^b V(\chi(t) + \lambda - \xi, t - h) dB(\xi)$$
$$= \int_h^y dt \cdot \int_a^b V(\chi(t) - \xi, t - h) dB(\xi).$$

The limit function is absolutely continuous on $h \leq y \leq g$. (λ is assumed to approach zero either through positive or through negative values.)

We shall first show that $\int_h^y | V | dt$ is uniformly bounded. From the identity

$$(a-b)\exp[-(a-b)^2] = (a-c)\exp[-(a-b)^2] + (c-b)$$
$$\times \exp[-(c-b)^2]\exp[-2(c-b)(a-c)]\exp[-(a-c)^2]$$

and the fact that $(\chi(t) - \chi(h))/(t-h)$ is bounded, we see that

$$(7) \quad | V(\chi(t) + \lambda - \xi, t - h) | \leq M(t-h)^{-1/2} + M | V(\chi(h) + \lambda - \xi, t - h |,$$

where M is a constant depending on $\chi(t)$ and $\chi'(t)$. From (7) it follows that the integral

$$(8) \qquad \int_h^y | V(\chi(t) + \lambda - \xi, t - h) | dt$$

is uniformly bounded for a given $\chi(y)$.

From Lemma 1 and a property of Young-Stieltjes integrals we may write

$$\lim_{\lambda \to 0} \int_h^y dt \cdot \int_a^b V(\chi(t) + \lambda - \xi, t - h) dB(\xi)$$
$$= \int_a^b dB(\xi) \cdot \lim_{\lambda \to 0} \int_h^y V(\chi(t) + \lambda - \xi, t - h) dt,$$

provided the limit of the integrand exists. For $\xi \neq \chi(h)$ and λ sufficiently small, V is continuous; this gives us the first of the following results:

$$\lim_{\lambda\to 0\pm}\int_h^y V(\chi(t)+\lambda-\xi,t-h)\,dt=\int_h^y V(\chi(t)-\xi,t-h)\,dt,\ \xi\neq\chi(h),$$

$$=\pm 1+\int_h^y V(\chi(t)-\xi,t-h)\,d\xi,\ \xi=\chi(h).$$

The second result, when $\xi=\chi(h)$, is well known.[8] Thus the limit of the Stieltjes integrand exists; in fact it is continuous in ξ except for $\xi=\chi(h)$. Since $B(\xi)$ is continuous at $\chi(h)$, the ± 1 contributes nothing, and we obtain

$$\lim_{\lambda\to 0}\int_h^y dt\cdot\int_a^b V(\chi(t)+\lambda-\xi,t-h)\,dB(\xi)$$

$$=\int_a^b dB(\xi)\int_h^y V(\chi(t)-\xi,t-h)\,dt\equiv\phi(y).$$

From (8) we may interchange the order of integration in $\phi(y)$. Again using (8), we then see that $\phi(y)$ is the indefinite integral of a summable function; hence $\phi(y)$ is an absolutely continuous function. This completes the proof of the theorem.

The function $H(\chi(y),y)$ appearing in Theorem 4 of paper (A) is a Stieltjes integral; we now proceed to change it into an ordinary integral.

$$H(x,y)=\int_h^y dG(\eta)\cdot\int_\eta^y 2\chi'(t)U(\chi(t)+\lambda-\chi(\eta),t-\eta)\,dt,$$

$$x=\lambda+\chi(y),\ \lambda\neq 0.$$

We may replace $G(\eta)$ by $\bar{G}(\eta)=G(\eta)-G(h)$ without changing the value of $H(x,y)$. We assume this done, and we also assume $\chi''(t)$ continuous. After integrating by parts, we can write H in the form

$$H(x,y)=-\int_h^y \bar{G}(\eta)\int_\eta^y\frac{\partial}{\partial\eta}[2\chi'(t)U]\,dt\cdot d\eta.$$

On adding and subtracting $\frac{\partial}{\partial t}[2\chi'(\eta)U]$ to the integrand of the inside integral, we obtain

$$(9)\quad H(x,y)=-\int_h^y\bar{G}(\eta)\int_\eta^y\left[\frac{\chi'(t)-\chi'(\eta)}{t-\eta}\right]\left[1-\frac{(\chi(t)+\lambda-\chi(\eta))^2}{2(t-\eta)}\right]$$

$$\cdot U(\chi(t)+\lambda-\chi(n),t-\eta)\,dt\,d\eta+\int_h^y 2\chi'(\eta)U(\chi(y)+\lambda-\chi(\eta),\ y-\eta)\bar{G}(\eta)$$

Since χ'' is continuous and since there exists a constant K such that

$$|\alpha|\exp(-|\alpha|)<K\exp(-|\alpha|/2),$$

the $\lim_{\lambda\to 0}H(x,y)$ may be obtained by putting $\lambda=0$ in (9).

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[8] E. Goursat, *Cours d'Analyse*, vol. 3, Paris (1927), p. 307.

To simplify the notation in the next theorem, define

$$(10)\quad R(\phi(y),\psi(\eta)) = V(\phi(y)-\psi(\eta), y-\eta)$$
$$-\int_\eta^y \left[\frac{\phi'(t)-\psi'(\eta)}{t-\eta}\right]\left[1-\frac{(\phi(t)-\psi(\eta))^2}{2(t-\eta)}\right] U(\phi(t)-\psi(\eta), t-\eta)\,dt.$$

THEOREM 4. *Let χ'' be continuous and $G(y)$ be of limited variation on $h \le y \le g$. Then, for each fixed y in $h < y \le g$, we can represent the following limit in two forms:*

$$\lim_{\lambda\to 0^+} \int_h^y dt \int_h^t V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dG(\eta)$$

is either

$$(11)\quad = \pm\,[G(y-0)-G(h)] + \int_h^y R(\chi(y),\chi(\eta))\,\bar{G}(\eta)\,d\eta,$$

or

$$(12)\quad = \pm\,[G(y-0)-G(h)] + \int_h^y dG(\eta)\cdot\int_\eta^y V(\chi(t)-\chi(\eta), t-\eta)\,dt.$$

The representation (11) follows from combining (9) for $\lambda = 0$ with the results of Theorem 4 of paper (A). We now proceed to prove (12). For $\lambda \neq 0$, we may change the order of integration and write

$$(13)\quad \lim_{\lambda\to 0^+} \int_h^y dG(\eta)\cdot\int_\eta^y V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt$$
$$= \int_h^y dG(\eta)\lim_{\lambda\to 0^+}\int_\eta^y V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt.$$

From a theorem similar to one of Goursat's [9]

$$\lim_{\lambda\to 0^+}\int_\eta^y V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt = \pm 1 + \int_\eta^y V(\chi(t)-\chi(\eta), t-\eta)\,dt, \qquad y \neq \eta,$$
$$= 0, \qquad y = \eta.$$

Putting the above result in (13), we see that (12) is true.

3. A boundary value problem. Consider the domain D defined in the introduction. We shall refer to the segments of the line $y = h+\delta$, $\delta > 0$, with end points $(x_1, h+\delta)$ and $(x_2, h+\delta)$ as admissible δ-displacements, if for sufficiently small δ the following set of points (x, y) lies in D plus its boundary

$$x_1 \le x \le x_2, \qquad h \le y \le h+\delta.$$

[9] *Loc. cit.*, p. 307.

The curves

$$x = \chi_i(y) + \lambda_i, \qquad (-1)^{i+1}\lambda_i > 0, \qquad h \leq y \leq g,$$

will be called λ-displacements if they lie in D except for their end points, which lie on the boundary of D. We shall show how to find a solution $u(x, y)$ of the heat equation (1) in D which for admissible δ-displacements satisfies relation (2), where $B(x)$ is a given function of bounded variation with regular discontinuities and continuous at $x = a$ and $x = b$. Moreover, for λ-displacements, $u(x, y)$ satisfies (3), where the $G_i(y)$ are preassigned functions of limited variation continuous from the left and continuous at $\chi_i(h)$.

Set up the function

$$(14)\quad u(x,y) = \int_a^b U(x-\xi, y-\eta)\,dB(\xi) - \sum_{i=1}^{2} \int_h^y 2U(x-\chi_i(\eta), y-\eta)\,dF_i(\eta)$$
$$= u_1(x,y) - u_2(x,y) - u_3(x,y).$$

We see that $u(x, y)$ satisfies (1) in D if the $F_i(\eta)$ are functions of limited variation. From Theorems 1 and 2, we have for admissible δ-displacements

$$\lim_{\delta \to 0} \int_{x_1}^{x_2} u(t, h+\delta)\,dt = B(x_2) - B(x_1),$$

if the $F_i(\eta)$ are continuous at $\eta = h$. For $u(x, y)$ to satisfy all our conditions, we have thus to show the existence of functions $F_i(\eta)$ of limited variation, continuous from the left and continuous at $\eta = h$, and such that

$$(15)\qquad \lim_{\lambda_i \to 0} \int_{y_1}^{y_2} u_x(\chi_i(t) + \lambda_i, t)\,dt = G_i(y_2) - G_i(y_1)$$

for λ-displacements.

We shall assume the $\chi''_i(y)$ continuous, and since $B(\xi)$ is continuous at $\xi = a$ and $\xi = b$, we have from Theorem 3

$$\lim_{\lambda_i \to 0} \int_b^y \frac{\partial}{\partial x} u_1(\chi_i(t) + \lambda_i, t)\,dt = \phi_i(y),$$

where the $\phi_i(y)$ are absolutely continuous functions of y.

Putting $y_1 = h$ and $y_2 = y$ and making use of result (11) in Theorem 4, the pair of equations (15) can be put in the form

$$(16)\qquad f_i(y) = g_i(y) + \sum_{j=1}^{2} \int_h^y K_{ij}(y, \eta) f_j(\eta)\,d\eta,$$

where the $g_i(y)$ are known functions of limited variation continuous from the left and continuous at $y = h$.

$$g_i(y) = (-1)^{i+1}[G_i(y) - G_i(h) - \phi_i(y)]$$

and

$$f_i(y) = F_i(y) - F_i(h).$$

$$K_{ij}(y, \eta) = (-1)^i R(\chi_i(y), \chi_j(\eta)), \tag{17}$$

the function R being given by (10).

Since the kernel $K_{ij}(y, \eta)$ is of the form of a continuous function of y and η divided by $(y - \eta)^{\frac{1}{2}}$, the system (16) will have a unique solution continuous from the left and continuous at $y = h$. It remains to show that the $f_i(y)$ are of limited variation. To show that the $f_i(y)$ are indeed of limited variation we proceed as we did in section 5 of paper (A). The only step we must supply here is to show that there exist functions $T_{ij}(y, \eta)$ such that

$$\int_\eta^y K_{ij}(y, \xi)\, d\xi = \int_\eta^y T_{ij}(\xi, \eta)\, d\xi.$$

Comparing the results (11) and (12) of Theorem 4, we see that we can take the

$$T_{ij}(y, \eta) = V(\chi_i(y) - \chi_j(\eta), y - \eta). \tag{18}$$

We collect the principal result of Part 1 into the following theorem:

THEOREM 5. *Let $B(x)$ be of limited variation for $\chi_1(h) \equiv a \leq x \leq b \equiv \chi_2(h)$, continuous at a and b, and having regular discontinuities for $a < x < b$. Let $G_1(y)$, $G_2(y)$ be of limited variation on $h \leq y \leq g$, continuous at h, and continuous from the left for $h < y \leq g$. Let χ_1 and χ_2 have continuous second derivatives. Then in the class of functions* (14), *there exists a unique function $u(x, y)$ satisfying the heat equation* (1) *in D and satisfying the boundary conditions* (2) *and* (3).

4. A note on paper (A). In our first paper the function $l(x, y)$ solving the boundary value problem was written

$$l(x, y) = \sum_{i=1}^{2} \int_h^y [V_i - 2\chi_i(\eta) U] dF_i(\eta) + \int_a^b U dB(\xi).$$

We wish to point out that by the use of Theorem 4 of this paper we could drop out the $\chi_i(\eta) U$ and write

$$l(x, y) = \sum_{i=1}^{2} \int_h^y V_i dF_i(\eta) + \int_a^b U dB(\xi). \tag{19}$$

This simplifies $l(x, y)$. The kernels of the integral equations that (19) leads

to are the same as the functions K_{ij} of (17). These kernels are not simpler in form than the kernels of paper (A). However, the T_{ij} then become those defined in (18) and are simpler than those appearing in our first paper.

PART 2

In this part of the paper we return to the classical problem of finding a solution of the heat equation (1) in the region D which takes on preassigned values on the boundary of D. Here, however, our boundary values are not continuous but are functions of limited variation. The approaches to the boundary are in a restricted sense that we describe in the next section. The Riemann-Stieltjes integral will suffice for the work of this part of the paper.

5. Definitions and notation. A point $P(x, y)$ lying in the region D will be said to approach a point $M(x_0, h)$ on the characteristic $y = h$ in the parabolic sense if

$$\lim_{P \to M} (x - x_0)/(y - h)^{\frac{1}{2}} = 0. \tag{20}$$

When a point P approaches a point $Q(\chi_i(y_0), y_0)$ through a set of points E of D, we divide E into two sets as follows: E_1 contains those points of E such that $y > y_0$, and $E_2 = E - E_1$. If Q is not a limit point of E_1, or if Q is a limit point of E_1 and in addition

$$\lim_{\substack{P \to Q \\ P \subset E_1}} |x - \chi_i(y_0)|/(y - y_0)^{\frac{1}{2}} = \infty, \tag{21}$$

we shall say that P approaches the point Q in the parabolic sense.[10]

Let $W(x, y)$ denote the function

$$W(x, y) = \frac{1}{\sqrt{\pi}} \int_{x/2\sqrt{y}}^{\infty} \exp(-u^2)\, du, \; y > 0; \qquad = 0, \; y \leq 0. \tag{22}$$

The following functions defined in terms of W are solutions of the heat equation (1) for points in D, and will play a fundamental rôle in the rest of the paper:

$$u(x, y) = \int_a^b [\tfrac{1}{2} - W(x - \xi, y - h)]\, dF(\xi), \tag{23}$$

$$v_i(x, y) = 2 \int_h^y W((-1)^{i+1}(x - \chi_i(\eta)), y - \eta)\, dG(\eta), \; x \neq \chi_i(y), \tag{24}$$

$$= 2 \int_h^y [W((-1)^{i+1}(x - \chi_i(\eta)), y - \eta) - W(0, 1)]\, dG(\eta), \; x = \chi_i(y).$$

[10] The above definitions of parabolic approaches are radically different, and it is to be noted that a corner point $(\chi_i(h), h)$ can not be reached simultaneously under both definitions.

The functions $F(\xi)$ and $G(\eta)$ are functions of bounded variation and $a = \chi_1(h)$, $b = \chi_2(h)$.

6. Parabolic approaches. In this section we shall be concerned with the limits of u and v_i under parabolic approaches to the boundary of D.

THEOREM 6. *If $P(x, y)$ approaches the point $M(x_0, h)$ $(a \leq x_0 \leq b)$ in the parabolic sense, then the function $u(P) \equiv u(x, y)$ defined by (23) has the following limit*

$$\lim_{P \to M} u(P) = \frac{F(x_0 + 0) + F(x_0 - 0)}{2} - \frac{F(b) + F(a)}{2},$$

where $F(b + 0) = F(b)$ and $F(a - 0) = F(a)$.

Suppose first that $a < x_0 < b$ and write $u(P)$ in the form

$$u(P) = \int_a^{x_0} [\tfrac{1}{2} - W(x - \xi, y - h)]\, dF(\xi) + \int_{x_0}^{b} [\tfrac{1}{2} - W(x - \xi, y - h)]\, dF(\xi)$$
$$= u_1(P) + u_2(P).$$

Integrating $u_1(P)$ by parts and replacing $F(\xi)\,d\xi$ by $d \int_a^{\xi} F(t)\,dt$, we may write

$$u_1(P) = [\tfrac{1}{2} - W(x - \xi, y - h)]F(\xi)\Big]_a^{x_0} + \int_a^{x_0} U(x - \xi, y - h)\, d\xi \int_a^{\xi} F(t)\,dt.$$

Under a parabolic approach the integrated part has the limit $-F(a)/2$. Since $\int_a^{\xi} F(t)\,dt$ has a left-sided derivative at x_0 equal to $F(x_0 - 0)$, Theorem 1 of paper (A) shows that the above integral in $u_1(P)$ has the limit $F(x_0 - 0)/2$. The limit of $u_1(P)$ is thus seen to be

$$\lim_{P \to M} u_1(P) = \frac{F(x_0 - 0)}{2} - \frac{F(a)}{2}.$$

Noting that $\int_a^{\xi} F(t)\,dt$ has a right-sided derivative at x_0 equal to $F(x + 0)$, the same type of argument shows that

$$\lim_{P \to M} u_2(P) = \frac{F(x_0 + 0)}{2} - \frac{F(b)}{2}.$$

From the above limits the theorem readily follows for $a < x_0 < b$. If x_0 is a or b then $u_1(P)$ or $u_2(P)$ is zero and no new considerations are needed in these cases.

Turning now to the functions $v_i(P)$, we see that as the point P approaches the point $M(x_0, h)$ with $a < x_0 < b$ these functions each have the

limit zero. Our next theorem is concerned with the limit of $v_i(P)$ as P approaches the point $Q(\chi_i(y_0), y_0)$.

THEOREM 7. *If P approaches Q in the parabolic sense, then the functions $v_i(P)$ defined by (24) have the limit*

$$\lim_{P\to Q} v_i(P) = [G(y_0 - 0) - G(h)] + v_i(\chi_i(y_0), y_0).$$

Let us consider the case of $v_1(P)$, and to simplify the notation write $v(P) \equiv v_1(P)$ and $\chi(\eta) \equiv \chi_i(\eta)$. It is then to be noted that $x - \chi(y) > 0$ for P in the region D.

For a given $\delta > 0$ we can select $\epsilon > 0$ so that the total variation of $G(\eta - 0)$ $[G(\eta + 0)]$ in $y_0 - \epsilon \leq \eta \leq y_0$ $[y_0 \leq \eta \leq y_0 + \epsilon]$ is less than δ. With ϵ fixed, since $y \to y_0$, we assume $y_0 - \epsilon/2 \leq y \leq y_0 + \epsilon/2$, and write $v(P)$ in the following form

$$\begin{aligned} v(P) &= \int_h^{y_0-\epsilon} 2W(x - \chi(\eta), y - \eta)\, dG(\eta - 0) + \int_{y_0-\epsilon}^{y} 2W\, dG(\eta - 0) \\ &= I_1(x, y) + I_2(x, y). \end{aligned}$$

Since $G(h - 0) = G(h)$ and the integrand W is continuous and equal to zero for $y = \eta$, the replacement of $G(\eta)$ by $G(\eta - 0)$ is permissible.

In I_1 the difference $y - \eta$ is bounded away from zero, hence

$$\lim_{P\to Q} I_1(x, y) = I_1(\chi(y_0), y_0).$$

For $y \leq y_0$, we have

$$|I_2(x, y)| < 2\delta.$$

If $y > y_0$, we write $I_2(x, y)$ as the sum of two integrals

$$I_2(x, y) = \left\{ \int_{y_0+\epsilon}^{y_0} + \int_{y_0}^{y} \right\} W(x - \chi(\eta), y - \eta)\, dG(\eta - 0).$$

The first integral is seen to be less than 2δ, while the second can be put in the form

$$\int_{y_0}^{y} W(x - \chi(\eta), y - \eta)\, dG(\eta + 0) + [G(y_0 + 0) - G(y_0 - 0)]\ W(x - \chi(y_0), y-$$

Here again the first term is less than 2δ, while the second has the limit zero when P approaches Q in the parabolic sense. Thus we see that

$$\lim_{P\to Q} |I_2| < 4\delta.$$

Since δ is any positive number, and since

$$\left| \int_{y_0-\epsilon}^{y_0} 2W(\chi(y_0)-\chi(\eta), y_0-\eta)\,dG(\eta-0) \right| < 2\delta,$$

we have

$$\lim_{Q} v(P) = \int_h^{y_0} 2W(\chi(y_0-\chi(\eta), y_0-\eta)\,dG(\eta-0)$$

$$= \int_h^{y_0} 2W(0,1)\,dG(\eta-0) + \int_h^{y_0} 2[W(\chi(y_0)-\chi(\eta), y_0-\eta) - W(0,1)]\,dG(\eta-0)$$

$$= G(y_0-0) - G(h) + v(\chi(y_0), y_0).$$

Theorem 7 is thus seen to hold for $v(P) = v_1(P)$ and a similar proof holds for $v_2(P)$.

7. The first boundary value problem. We shall prove the existence of a function $l(x,y)$ which is a solution of (1) in D, and whose limit as the point $P(x,y)$ approaches the point $M(x_0,h)$ in the parabolic sense is equal to a given function $B(x_0)$ of limited variation with regular discontinuities [here $a = \chi_1(h) < x_0 < \chi_2(h) = b$], also the limit of $l(P)$ as P approaches in the parabolic sense a point $Q(\chi_i(y_0), y_0)$ with $y_0 > h$ is equal to a given function $H_i(y_0)$ of limited variation continuous from the left.

Let $F(x) = B(x)$ for $a < x < b$ and let $F(a) = F(b) = 0$, and set up the function

$$(25) \qquad l(x,y) = \sum_{i=1}^{2} 2 \int_h^y W((-1)^{i+1}[x-\chi_i(\eta)], y-\eta)\,dG_i(\eta)$$

$$+ \int_a^b [\tfrac{1}{2} - W(x-\xi, y-h)]\,dF(\xi)$$

$$= v_1(x,y) + v_2(x,y) + u(x,y).$$

We see that in the domain D the function $l(x,y)$ satisfies equation (1) if the $G_i(\eta)$ are functions of limited variation. Since the $v_i(x,y)$ approach zero as P approaches $M(x_0,h)$ for $a < x_0 < b$, we see from Theorem 6 that under a parabolic approach

$$(26) \qquad \lim_{P\to M} l(x,y) = B(x_0).$$

As P approaches $Q(\chi_i(y_0), y_0)$ in the parabolic sense with $y_0 > h$, we have from Theorem 7

$$(27) \qquad \lim_{P\to Q} l(x,y) = [G_i(y_0-0) - G_i(h)] + \sum_{j=1}^{2} v_j(\chi_i(y_0), y_0) + u(\chi_i(y_0), y),$$

and we must show the existence of functions of limited variation $G_i(y)$ so that the above limit is equal to the given functions:

$$(28) \qquad H_i(y_0).$$

Let us note that the $v_i(\chi_j(y), y)$ approach zero with y approaching h when $i = j$ and also when $i \neq j$. Also as $(\chi_i(y) - \chi_i(h))/(y - h)^{\frac{1}{2}}$ approaches zero with $y - h$, we see from Theorem 6 that $\lim u(\chi_i(y), y)$ as y approaches h exists and has a finite value. Taking this value for $u(\chi_i(h), h)$ the function $u(\chi_i(y), y)$ is a continuous function of y in $h \leq y \leq g$.

Set

$$(29)\quad f_i(y) = G_i(y-0) - G_i(h) + u(\chi_i(h), h) - H_i(h),\ y > h;\ = 0,\ y = h,$$

and on equating (28) to the right member of (27) with $y_0 = y$, we have the following system of integral equations

$$(30)\qquad f_i(y) = g_i(y) + \sum_{j=1}^{2} \int_h^y K_{ij}(y, \eta)\, df_j(\eta), \qquad (i = i, 2),$$

where [11]

$$g_i(y) = [H_i(y) - H_i(h)] - [u(\chi_i(y), y) - u(\chi_i(h), h)],$$

$$(31)\quad K_{ij}(y, \eta) = -2[W((-1)^{j+1}[\chi_i(y) - \chi_j(\eta)], y - \eta) - \delta_j{}^i W(0, 1)].$$

Here $\delta_j{}^i$ is 1 or 0 according as $i = j$ or $i \neq j$. The kernels K_{ij} can be put into the two integral forms

$$(32)\quad K_{ij}(y, \eta) = (-1)^j \int_\eta^y \left[\frac{\chi_i(y) - \chi_j(t)}{t - \eta} - 2\chi'_j(t)\right] U(\chi_i(y) - \chi_j(t), y - t)$$

$$(33)\qquad = (-1)^j \int_\eta^y \left[\frac{\chi_i(t) - \chi_j(\eta)}{t - \eta} - 2\chi'_i(t)\right] U(\chi_i(t) - \chi_j(\eta), t - \eta)$$

We have seen [12] that the system of integral equations (30) having kernels with the representations (32)-(33) and with $g_i(h) = 0$, have unique solutions $f_i(y)$ of limited variation and continuous from the left provided that the $g_i(y)$ are such functions. The $H_i(y)$ have these properties and the $u(\chi_i(y), y)$ are continuous; it remains to show that the $u(\chi_i(y), y)$ are of limited variation. We shall do this in the last section. Thus we state the result:

LEMMA 2. *For a given set of functions $g_i(y)$ of limited variation, continuous from the left, and vanishing at h, there exists a unique set of functions $f_i(y)$ satisfying* (30) *and having these same properties.*

The equations (29) determine each $G_i(y - 0)$ to within an additive constant which does not affect the function $l(x, y)$; moreover, $l(x, y)$ is in-

[11] The constant $u(\chi_i(h), h) - H_i(h)$ is put in to make $g_i(h) = 0$.

[12] Paper (A), pp. 651-653.

variant under the replacement of $G_i(y)$ by $G_i(y-0)$. Considering then the $g_i(y)$, we see that if $u(x,y)$ is fixed the $g_i(y)$ are unique. If we put on the restriction

$$F(a) = F(b) \tag{34}$$

in forming the function $u(x,y)$ defined in equation (23), it then follows from Theorem 6 that the only $u(x,y)$ which, under a parabolic approach to $M(x_0,h)$ $(a<x_0<b)$, has the limiting value zero, is $u(x,y)$ identically zero. We shall say that $u(x,y)$ is regular if it is formed under the restriction (34).

We now state the main theorem of Part 2:

THEOREM 8. *In the class of functions* (25) *with* $u(x,y)$ *regular, there exists a unique function* $l(x,y)$ *such that* $l(x,y)$ *is a solution of* (1) *in* D *and, under parabolic approaches to the boundary, takes on preassigned functions of limited variation. On the characteristic* $y=h$ *the assigned function has regular discontinuities, elsewhere the assigned functions are continuous from the left.* [*Each* $\chi'_i(y)$ *is assumed continuous.*]

8. Functions of limited variation. In this section we shall show that the functions $u(\chi_i(y),y)$ appearing in the integral equations (30) are of bounded variation.

Let $\chi(y)$ have a continuous derivative and define

$$H(y) = \int_a^b \int_\xi^{\chi(y)} U(t-\xi, y-h)\,dt\,dF(\xi), \qquad h<y\le g, \tag{35}$$

$$= \lim_{y\to h} \int_a^b \int_\xi^{\chi(y)} U(t-\xi, y-h)\,dt\,dF(\xi), \qquad y=h.$$

$H(y)$ is seen to be equal to the function $u(\chi(y),y)$ and Theorem 1 insures the existence of the above limit.

LEMMA 3. *The function* $H(y)$ *defined by* (35) *is of limited variation in* $h\le y\le g$.

We give the proof for $\chi(h)=b$; the proof for other values of $\chi(h)$ can be handled in a similar manner.

Write $H(y)$ in the form

$$H(y) = \int_a^b \int_\xi^b U\,dt\,dF(\xi) + \int_a^b \int_b^{\chi(y)} U\,dt\,dF(\xi), \qquad y>h$$

$$= I_1(y) + I_2(y).$$

Under the transformation $(t-\xi)/2(y-h)^{\frac{1}{2}}$ the function

$$\int_{\xi}^{b} U(t-\xi, y-h)\,dt$$

is seen to be monotonically decreasing with y for each ξ, and to have a total variation less than one on the set $y > h$. It then follows that the variation of $I_1(y)$ on this set is less than the total variation of $F(\xi)$.

With $y_2 \geq y_1 > h$, write

$$I_2(y_2) - I_2(y_1) = \int_a^b \int_{\chi(y_1)}^{\chi(y_2)} U(t-\xi, y_2-h)\,dt\,dF(\xi)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_a^b \int_b^{\chi(y_1)} \left[\frac{1}{\sqrt{y_2-h}} - \frac{1}{\sqrt{y_1-h}}\right] \exp \frac{(t-\xi)^2}{4(y_1-h)}\,dt\,dF(\xi)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_a^b \int_b^{\chi(y_1)} \frac{1}{\sqrt{y_2-h}} \left[\exp\left(-\frac{(t-\xi)^2}{4(y_2-h)}\right) - \exp\left(-\frac{(t-\xi)^2}{4(y_1-h)}\right)\right] dt\,dF$$

Remembering that $b = \chi(h)$ and that $\chi'(\eta)$ is bounded, the first two integrals on the right are easily shown to be less than $\sqrt{y_2-h} - \sqrt{y_1-h}$ times a constant C. The constant C depends only on the maximum of $\chi'(\eta)$ and the total variation of $F(\xi)$. We obtain for the third integral a similar bound, if, after using the theorem of the mean, the following property of $\exp(-\alpha)$ is used. If P, k are two positive constants with $0 < k < 1$ and α is a variable ≥ 0, then there exists a constant K such that

$$\alpha^P \exp(-\alpha) < K \exp(-k\alpha).$$

Since the function $\sqrt{y-h}$ is of limited variation, the function $I_2(y)$ is seen to have a variation on $h < y \leq g$ less than a constant times the variation of $\sqrt{y-h}$.

As the function $H(y)$ is bounded it then follows that it is of limited variation on the closed interval $h \leq y \leq g$.

DUKE UNIVERSITY AND
HAMPDEN SYDNEY COLLEGE.

ANALYTIC CONTINUATION OF FUNCTIONS DEFINED BY FACTORIAL SERIES.*

By GLEN T. MILLER and HOWARD K. HUGHES.

In this paper we shall be concerned with the determination of an analytic continuation of a function $\Omega(z)$ defined by a factorial series of the form:

$$\Omega(z) = \sum_{m=0}^{\infty} \frac{g(m)(-1)^{m(k+1)}}{z(z+1)\cdots(z+m)}, \tag{1}$$

where k is a positive integer, and z is a complex variable.

The results here obtained may be considered to be an extension of a theorem obtained by Hughes [1] for a function defined by a series of the type (1) when $k = 0$. The methods used bear a close analogy to those used by Ford [2] and Newsom [3] in their work on functions defined by Maclaurin series.

It is well known that the region of convergence of a factorial series is a half-plane bounded on the left by a vertical line $R(z) = \lambda$, where $R(z)$ denotes the real part of z and $-\infty \leqq \lambda \leqq \infty$. Furthermore the convergence is uniform in the half-plane $R(z) = \lambda + \delta$ where δ is any positive number. Consequently the series (1) represents an analytic function in the region $R(z) > \lambda$.[4]

The results are given in the following

THEOREM. *Given the factorial series* (1), *whose abscissa of convergence is finite, and for which* k *is a positive integer; let it be assumed that the function* $g(m)$ *occurring in the coefficient of* (1), *when considered as a function* $g(w)$ *of the complex variable* $w = x + iy$, *satisfies the following conditions*:

(a) *it is single-valued and analytic throughout the half-plane* $x \geqq -\frac{1}{2}$;

* Received January 15, 1942.

[1] H. K. Hughes, "On the analytical extension of functions defined by factorial series," *American Journal of Mathematics*, vol. 53 (1931), pp. 771-776.

[2] W. B. Ford, "The asymptotic developments of functions defined by Maclaurin series," *University of Michigan Studies*, Scientific Series, Vol. 11.

[3] C. V. Newsom, "On the character of certain entire functions in distant portions of the plane," *American Journal of Mathematics*, vol. 60 (1938), pp. 561-572.

[4] For an account of the properties of factorial series, see *The Calculus of Finite Differences* by Milne-Thomson (1933), Ch. X.

(b) *for all* $x \geqq -\frac{1}{2}$, *and for all values of* $|\, y \,|$ *sufficiently large, we have:*

$$\left| \frac{g(x+iy)}{g(x)} \right| < M \exp[\{\pi(k+\tfrac{1}{2}) - \epsilon\} |\, y \,|] \tag{2}$$

where ϵ *is an arbitrarily small positive number and* M *is a constant dependent on* ϵ.

Moreover, let there exist a positive constant q, *dependent on* ϵ, *but independent of* z, *such that when* z *is confined to any finite region, we have,*

$$\left| \frac{g(x)}{\Gamma(z+x+1)} \right| < \frac{N}{x^{1+\epsilon}} \tag{3}$$

when $x > q$, *and where* N *is a positive constant.*

Then the function $\Omega(z)$, *defined by* (1) *may be continued analytically throughout the whole finite plane, except in the neighborhoods of the points* $z = 0, -1, -2, \cdots$, *and throughout this region will be defined by the equation*

$$\Omega(z) = \Gamma(z) \int_{-\frac{1}{2}}^{\infty} \frac{\sin k\pi x}{\sin \pi x} \frac{g(x)}{\Gamma(z+x+1)}\, dx + \Gamma(z)(-\tfrac{1}{2}i)^k k\pi \int_{-\infty}^{+\infty} \frac{F(y,z)}{(\cosh \pi y)^{k+1}}\, dy \tag{4}$$

where

$$F(y,z) = \int_0^y \frac{[\sinh \pi(s-y)]^{k-1} g(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\, ds. \tag{5}$$

The proof of this theorem is based on the following theorem in the Calculus of residues, due to Newsom:

"Let $P(w)$ be regular within a closed region bounded by the curve C in the w-complex plane, and enclosing the points $m = 0, 1, 2, \cdots, n$, but no other vanishing points of $\sin \pi w$. Furthermore, let $P(m) \neq 0$, $m = 0, 1, 2, \cdots, n$. Then

$$\frac{1}{2\pi i} \int_C \frac{P(w)}{(\sin \pi w)^{k+1}}\, dw = \frac{(-1)^{m(k+1)}}{\pi^{k+1} k!} \sum_{t=0}^{k} \pi^t (\Sigma \alpha_1 \alpha_2 \cdots \alpha_t) P^{(k-t)}(m) \tag{6}$$

where $m = 0, 1, 2, \cdots, n$, and where $(\Sigma \alpha_1 \alpha_2 \cdots \alpha_t)$ denotes the sum of the $\binom{k}{t}$ products of t factors each formed by taking the possible combinations of the k quantities $\pm (k-1)i, \pm (k-3)i, \cdots \left\{ \begin{matrix} \pm i \\ 0 \end{matrix} \right\}$ t at a time. Here $\left\{ \begin{matrix} \pm i \\ 0 \end{matrix} \right\}$ is $\pm i$ or 0 according as k is even or odd. The integration is performed in the positive sense." [5]

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[5] Newsom, *op. cit.*, pp. 562, 563.

Proof of the Theorem.

Let us suppose that z has any fixed value which is real and positive and greater than λ, the abscissa of convergence of series (1). We shall write the series (1) in the form:

$$\Omega(z) = \sum_{m=0}^{\infty} \frac{g(m)\Gamma(z)(-1)^{m(k+1)}}{\Gamma(z+m+1)} \tag{7}$$

and shall seek to determine $P(w)$ so that

$$\sum_{t=0}^{k} \pi^t (\Sigma\alpha_1\alpha_2 \cdots \alpha_t) P^{(k-t)}(m) = \frac{g(m)}{\Gamma(z+m+1)}. \tag{8}$$

This is a differential equation in m with constant coefficients. The roots of the characteristic equation are distinct, and are $\pm(k-1)\pi i$, $\pm(k-3)\pi i, \cdots \left\{ \begin{matrix} \pm\pi i \\ 0 \end{matrix} \right\}$. If we denote these roots by $r_j\pi i$, where

$$r_j = k+1-2j, \qquad 1 \leqq j \leqq k, \tag{9}$$

then a particular solution of (8) is given by

$$P(m) = \sum_{j=1}^{k} b_j \exp r_j\pi i m \int_0^m \frac{\exp(-r_j\pi i m) g(m)}{\Gamma(z+m+1)}\, dm \tag{10}$$

where

$$b_j = \frac{1}{(2\pi i)^{k-1}} \prod_{a=1}^{k} \frac{1}{a-j} \qquad a \neq j. \tag{11}$$

In some of our later considerations we shall find the following form for b_j useful:

$$b_j = \frac{1}{(2\pi i)^{k-1}(k-1)!} \binom{k-1}{j-1} (-1)^{j-1}. \tag{12}$$

If we replace m by the complex variable $w = x + iy$, then $P(w)$ is an analytic function of w in the finite w-plane. Hence if $P(w)$ has the form indicated by (10), then the equation (6) can be written in the form:

$$\sum_{m=0}^{n} \frac{(-1)^{m(k+1)} g(m)\Gamma(z)}{\Gamma(z+m+1)} = \frac{\Gamma(z)\pi^k k!}{2i} \int_C \frac{P(w)}{(\sin \pi w)^{k+1}}\, dw \tag{13}$$

where the contour C is so chosen as to enclose the points $w = 0, 1, 2, \cdots, n$, but no other vanishing points of $\sin w$.

If we substitute the form for $P(w)$ from (10) in (13) we obtain a contour integral for the determination of the first $n+1$ terms of the series (7) as follows:

$$(14)\qquad \sum_{m=0}^{n} \frac{(-1)^{m(k+1)} g(m)\Gamma(z)}{\Gamma(z+m+1)} = \frac{\Gamma(z)\pi^k k!}{2i} \sum_{j=1}^{k} b_j \int \frac{\exp r_j \pi i w \, I(w,z)}{(\sin \pi w)^{k+1}}\, dw$$

where

$$(15)\qquad I(w,z) = \int_0^w \frac{\exp(-r_j \pi i w) g(w)}{\Gamma(z+w+1)}\, dw,$$

and where b_j is given by (11).

We shall choose as the path of integration the rectangle enclosed by the lines (1) $w = x + ip$, (2) $w = -\frac{1}{2} + iy$, (3) $w = x - ip$, and (4) $w = n + \frac{1}{2} + iy$, and shall denote the integrals along these sides by A, B, D, and E, respectively. Then in order to evaluate the contour integral

$$\int_C \frac{P(w)}{(\sin \pi w)^{k+1}}\, dw$$

we must consider k integrals of the form:

$$(16)\qquad \int_C \frac{\exp r_j \pi i w \;\; I(w,z)}{(\sin \pi w)^{k+1}}\, dw$$

where $1 \leqq j \leqq k$, and $I(w,z)$ is given by (15).

In (15) let us take the path of integration along the x-axis from $w = 0$ to $w = x$, and then along a line parallel to the y-axis to the point $w = x + iy$. The function $I(w,z)$ then can be written in the form:

$$I(w,z) = R(x,z) + iS(x,y,z),$$

where

$$R(x,z) = \int_0^x \frac{\exp(-r_j \pi i x) \;\; g(x)}{\Gamma(z+x+1)}\, dx,$$

and

$$S(x,y,z) = \int_0^y \frac{\exp(-r_j \pi i (x+iy)) \;\; g(x+iy)}{\Gamma(z+x+1+iy)}\, dy.$$

With regard to $R(x,z)$ it is sufficient for our purpose to note that $|R(x,z)|$ is bounded when x is bounded.

It is known that for all $x > x_0$, where x_0 is arbitrary, and for all y sufficiently large, we have:[6]

$$(17)\qquad \left| \frac{1}{\Gamma(x+iy)} \right| < K \exp[\tfrac{1}{2}(\pi + \epsilon)|y|].$$

From the above inequality, together with (2) and (9), it follows that the absolute value of the integrand in (15) is less than

$$m_1 \exp[r_j \pi y + \{(k+1) - \tfrac{1}{2}\epsilon\}|y|],$$

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[6] Ford, *op. cit.*, p. 61.

the constant m_1 being dependent only on ϵ. It follows that there exists a constant K, dependent only upon ϵ, such that

$$|S(x,y,z)| < K\exp[r_j\pi y + \{(k+1) - \tfrac{1}{2}\epsilon\}|y|]. \tag{18}$$

We are now ready to consider in detail the various contributions to the integral (16). We shall first consider the contribution A. Here we have $w = x + ip$, and

$$\sin \pi w = \sin \pi p(\sin \pi x \coth \pi p + i\cos \pi x).$$

Since $\lim_{p\to\infty}(\sin \pi x \coth \pi p + i\cos \pi x) = 1$, $\sin \pi w$ is of the order of $\exp \pi p$ when p is large.

In order to evaluate A we must consider k integrals of the form:

$$A_j = \int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{\exp r_j\pi i(x+ip)[R(x,z) + iS(x,p,z)]}{\exp \pi p(k+1)}\,dx$$

where $1 \leqq j \leqq k$. It is evident that we may write

$$|A_j| \leqq \left|\int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{R(x,z)}{\exp \pi p(k+1+r_j)}\,dx\right| + \left|\int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{S(x,p,z)}{\exp \pi p(k+1+r_j)}\,dx\right|. \tag{19}$$

Since $R(x,z)$ is bounded if x is bounded, the first integral on the right of (19) approaches zero as p approaches infinity.

With regard to the second integral on the right of (19), we have from (18) that

$$|S(x,p,z)| < K\exp([\pi(r_j + k + 1) - \tfrac{1}{2}\epsilon]p).$$

The absolute value of the integrand in the integral in question is therefore of the order of $\exp(-\frac{1}{2}\epsilon p)$ for p large, and since ϵ is positive we conclude that $\lim_{p\to\infty} A_j = 0$. It follows that $\lim_{p\to\infty} A = 0$.

In a similar manner it may be shown that $\lim_{p\to\infty} D = 0$.

We next consider the contribution E. We shall choose n to be an even integer so that $\sin \pi w = \cosh \pi y$. E consists of k integrals of the form:

$$E_j = i^{r_j}\left[iR(n+\tfrac{1}{2},z)\int_{-\infty}^{+\infty} \frac{\exp(-r_j\pi y)}{(\cosh \pi y)^{k+1}}\,dy - \int_{-\infty}^{+\infty} \frac{\exp(-r_j\pi y)S(n+\tfrac{1}{2},y,z)}{(\cosh \pi y)^{k+1}}\,dy\right]. \tag{20}$$

We shall set

$$\int_{-\infty}^{+\infty} \frac{\exp(-r_j\pi y)}{(\cosh \pi y)^{k+1}}\,dy = H_j.$$

Then by using the result of Newsom [7] and (9) we can write

[7] Newsom, *op. cit.*, pp. 567, 568.

$$H_j = \frac{(-1)^j 2^k}{-j\pi} \prod_{c=1}^{k} (1 - j/c) \qquad c \neq j.$$

Let us next consider the second integral on the right of (20). The integral can be written:

$$(21) \qquad \int_{-\infty}^{+\infty} \frac{\exp(-r_j \pi y) S(n + \tfrac{1}{2}, y, z)}{(\cosh \pi y)^{k+1}} dy$$

$$= \left\{ \frac{g(n)}{(z + n + 1)} \right\} \int_{-\infty}^{+\infty} \frac{\exp(-r_j \pi y) S_1(n + \tfrac{1}{2}, y, z)}{(\cosh \pi y)^{k+1}} dy$$

where

$$S_1(n + \tfrac{1}{2}, y, z) = \int_0^y \frac{\Gamma(z + n + 1) g(n + \tfrac{1}{2} + iy) \exp r_j \pi y}{\Gamma(z + n + 3/2 + iy) g(n)} dy.$$

We now make use of the following well known property of the gamma function: [8]

$$\Gamma(x + iy) = (2\pi)^{\frac{1}{2}} (x + iy)^{x - \frac{1}{2}} \exp(-x - y \arctan y/x)(1 + \delta)$$

where δ approaches zero as either x or y becomes infinite. Then we have

$$(22) \qquad \left| \frac{\Gamma(z + n + 1)}{\Gamma(z + n + 3/2 + iy)} \right| < K \exp(\pi/2 \mid y \mid).$$

By the use of (22) and (2), and taking into account the behavior of $(\cosh \pi y)$ for $\mid y \mid$ large, it follows that the integrand of the right member of (21) is of the order of $\exp(-\epsilon \mid y \mid)$, for $\mid y \mid$ large, and hence the integral converges. Furthermore, since the series (1) was assumed convergent, we have:

$$\lim_{n \to \infty} \frac{g(n)}{\Gamma(z + n + 1)} = 0.$$

Consequently the integral on the left of (21) tends to zero as n approaches infinity. The total contribution E_j is given by

$$E_j = i^{r_j + 1} H_j \lim_{n \to \infty} R(n + \tfrac{1}{2}, z).$$

We next consider the contribution B arising from the integration along the side where $w = -\frac{1}{2} + iy$. Here $\sin \pi w = -\cosh \pi y$ and we must consider k integrals of the form:

$$(23) \quad B_j = \int_{+\infty}^{-\infty} \frac{\exp r_j \pi i(-\tfrac{1}{2} + iy) R(-\tfrac{1}{2}, z)}{(-\cosh \pi y)^{k+1}} dy$$

$$+ i \int_{+\infty}^{-\infty} \frac{\exp r_j \pi i(-\tfrac{1}{2} + iy) S(-\tfrac{1}{2}, y, z)}{(-\cosh \pi y)^{k+1}} dy.$$

[8] Ford, *op. cit.*, p. 61.

The first of the integrals above, when combined with the contribution E_j, can be written in the form:

$$i^{r_j+1}H_j \lim_{n\to\infty}[R(n+\tfrac{1}{2},z)-R(-\tfrac{1}{2},z)]=i^{r_j+1}H_j\int_{-\frac{1}{2}}^{\infty}\frac{\exp(-r_j\pi ix)}{\Gamma(z+x+1)}\,dx.$$

If we multiply the above integral by b_j, we obtain as the sum of the j integrals in question when substituted in (14)

$$\Gamma(z)\int_{-\frac{1}{2}}^{\infty}\frac{\sin k\pi x}{\sin\pi x}\,\frac{g(x)}{\Gamma(z+x+1)}\,dx. \tag{24}$$

Finally let us investigate the second integral in (23). When substituted in (14), the sum of the k integrals in question may be written in the form:

$$\frac{\Gamma(z)\pi k}{(-2i)^k}\int_{+\infty}^{-\infty}\frac{\sum_{j=1}^{k}\int_0^y\binom{k-1}{j-1}\frac{(-1)^{j-1}\exp r_j\pi(s-y)g(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\,ds}{(\cosh\pi y)^{k+1}}\,dy. \tag{25}$$

But

$$\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}\exp r_j\pi(s-y)=2^{k-1}(\sinh\pi(s-y))^{k+1}. \tag{26}$$

Upon substituting (26) in (25), we find the value of this contribution to be:

$$\Gamma(z)k\pi(-i/2)^k\int_{+\infty}^{-\infty}\frac{\int_0^y\frac{[\sinh\pi(s-y)]^{k-1}q(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\,ds}{(\cosh\pi y)^{k+1}}\,dy. \tag{27}$$

The value of the right member of (14) is therefore the sum of the expressions (24) and (27). We thus obtain equation (4).

Up to this point we have restricted z to values which are real, positive and greater than λ, the abscissa of convergence. $\Omega(z)$ as defined by series (1) is a function which is analytic throughout the half-plane $R(z)>\lambda$ with the exception of the points $z=0,-1,-2,\cdots$. The right member of (4) furnishes an analytic continuation of $\Omega(z)$ in any finite region which does not contain the points $z=0,-1,-2,\cdots$, since throughout such a region the member under discussion is an analytic function. To prove this last statement, it is sufficient to show that the integrals on the right of (4) converge uniformly for all values of z in the proposed region. From (3), we see that the first integral in the right member of (4) converges uniformly, since it is dominated by $\int^{\infty}x^{-1-\epsilon}dx$. From (2), (5), and (17), it is evident that the second integral converges uniformly, since it is dominated by $\int^{\infty}\exp(-\frac{1}{2}\epsilon y)\,dy$. The theorem is therefore proved.

Extensions of the Theorem.

In the hypotheses of the theorem it was required that the function $g(w)$ be analytic for all values of $w = x + iy$ for which $x \geqq -\frac{1}{2}$. This condition is unnecessarily restrictive, as we shall now show. Let us assume that $g(w)$ has a singularity at a point $w = w_p$ in this region, and that w_p is not an integer. We shall now show that the theorem continues to hold provided that we subtract from the right member of (4) the loop integral

$$J_p = \frac{\Gamma(z)}{2i} \int_{L_p} \frac{\exp k\pi i w \sin \pi w \ \Gamma(z+w+1)}{g(w)} \ dw$$

wherein the contour L_p is described as follows: Draw a straight line from $w = w_p$ to infinity in a direction of the fourth quadrant, and let this line be regarded as a cut in the w-plane. Let the contour L_p consist of the two lines drawn on either side of and arbitrarily close to the cut, the ends near the point $w = w_p$ being joined by a circular arc of small radius about the same point. The integration is to be performed in the positive sense. We note that J_p is analytic for all finite values of z except $z = 0, -1, -2, \cdots$.

An examination of the analysis of the proof of the theorem shows that if $g(w)$ has the singularity mentioned, then the theorem still holds if we subtract from the right member of (4) the loop integral

$$H_p = \frac{k!\,\pi^k \Gamma(z)}{2i} \int_{L_p} \frac{P(w)}{(\sin \pi w)^{k+1}} \ dw$$

where L_p denotes the contour just described. We shall show that H_p can be reduced to J_p.

Noting the value of $P(w)$ given by (10), it is evident that we may write $P(w)$ in the following form:

$$P(w) = \int_0^w \sum_{j=1}^{k} b_j \exp r_j \pi i (w-t) \frac{g(t)}{\Gamma(z+t+1)} \ dt.$$

By using the form of b_j given by (12) it is easily verified that

$$\sum_{j=1}^{k} b_j \exp r_j \pi i (w-t) = \frac{[\sin \pi(w-t)]^{k-1}}{\pi^{k-1}(k-1)!} .$$

Thus we have

$$H_p = \frac{k\pi \Gamma(z)}{2i} \int_{L_p} \frac{\int_0^w [\sin \pi(w-t)]^{k-1} \frac{g(t)}{\Gamma(z+t+1)} \ dt}{(\sin \pi w)^{k+1}} dw.$$

Let $\bar{w}$ be a point on the cut, and let L'_p be that part of L_p drawn from w_p to $\bar{w}$, on both sides of the cut. Let

$$H_p(\bar{w}) = \frac{k\pi\Gamma(z)}{2i} \int_{L'_p} \frac{\int_0^w [\sin \pi(w-t)]^{k-1} \frac{g(t)}{\Gamma(z+t+1)}\, dt}{(\sin \pi w)^{k+1}}\, dw.$$

If we use the multiple angle formula and expand $[\sin \pi(w-t)]^{k-1}$ by the binomial theorem, we obtain:

$$H_p(\bar{w}) = \frac{k\pi\Gamma(z)}{2i} \sum_{j=1}^{k} \binom{k-1}{j-1} \int_{L'_p} (-\cot nw)^{j-1} \csc^2 \pi w \int_0^w \frac{(\cos \pi t \sin nt)^{j-1} g(t)}{\Gamma(z+t+1)}\, dt\, dw.$$

After integrating by parts and making simple reductions, we obtain

$$H_p(\bar{w}) = \frac{k\Gamma(z)}{2i} \int_{L'_p} \frac{\sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{k-1}{j-1} [\cot n\bar{w} \sin \pi w]^j (\cos \pi w)^{k-j} - (\cos \pi w)^k}{\sin \pi w \Gamma(z+w+1)}\, dw.$$

If we carry out the summation under the integral sign and make simple transformations, we obtain

$$H_p(\bar{w}) = \frac{\Gamma(z)}{2i} \int_{L'_p} \frac{\left[\exp(-\pi i w) + \exp(-\pi i \bar{w}) \frac{\sin \pi w}{\sin \pi \bar{w}}\right]^k g(w)}{\sin \pi w\ \Gamma(z+w+1)}\, dw.$$

On setting

$$J_p(\bar{w}) = \frac{\Gamma(z)}{2i} \int_{L'_p} \frac{g(w)}{\exp k\pi i w \sin \pi w\ \Gamma(z+w+1)}\, dw$$

it is evident that $\lim_{w\to\infty} J_p(\bar{w}) = J_p$.

We shall show that as $\bar{w} \to \infty$ along the cut, we have $\lim_{\bar{w}\to\infty} [J_p(\bar{w}) - H_p(\bar{w})] = 0$, and it will follow that $\lim_{\bar{w}\to\infty} H_p(\bar{w}) = J_p$. Now we have:

$$J_p(\bar{w}) - H_p(\bar{w}) = \Gamma(z) \exp(-\pi i w) \times \int_{L'_p} \frac{\sum_{m=1}^{k} \binom{k}{m} \exp(-\pi i w)(k-m) \cdot \exp(-\pi i w)(m-1) \left(\frac{\sin \pi w}{\sin \pi \bar{w}}\right)^m g(w)}{\sin \pi w\ \Gamma(z+w+1)}\, dw.$$

But if $\bar{w} \to \infty$ in the fourth quadrant we have for $\bar{w}$ large and for all w on L'_p $\left|\frac{\sin \pi w}{\sin \pi \bar{w}}\right| < 1$ and $\exp(-\pi i \bar{w}) < \exp(-\pi i w)$ and hence

$$|J_p(w) - H_p(w)| < \left|\frac{\Gamma(z)}{2} \exp(-\pi i \bar{w})\right| 2^k \int_{L_p} \frac{g(w)}{\exp(k-1)\pi w \sin \pi w\, \Gamma(z+w+1)}\, dw.$$

The integral exists because of the hypotheses of the theorem, and since the factor $\exp(-\pi i\bar{w})$ approaches zero, the above quantity can be made arbitrarily small. Hence if the function $g(w)$ has a finite number of singularities at the w_p, $p = 1, 2, \cdots, n$, the theorem continues to hold if we subtract from the right member of (4) the sum of the n integrals J_p. We note that if the singularity of $g(w)$ at W_p is a pole, then the integral J_p reduces to the residue of the function

$$\Gamma(z)^{k+1}\pi k! \ \frac{P(w)}{(\sin \pi w)^{k+1}}$$

where $w = w_p$ and $P(w)$ is given by (10).

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A KIND OF EVEN-DIMENSIONAL DIFFERENTIAL GEOMETRY AND ITS APPLICATION TO EXTERIOR CALCULUS.*

By HWA-CHUNG LEE.[1]

Introduction. In this paper, we study the differential geometry of an even-dimensional space, into which a nonsingular skewsymmetric matrix is introduced as fundamental tensor. In this geometry we define a curvature tensor from which there arises the distinction between general, flat and conformally flat spaces. Necessary and sufficient conditions for such spaces to be flat or conformally flat are found in **1, 2**. The results are applied, in **3**, to the exterior calculus to discover a necessary and sufficient condition that a nonsingular exterior differential form of degree two admit an integrating factor. The paper is concluded in **4** with a brief study of the infinitesimal automorphisms of the spaces.

1. The space L_{2n}. Consider a manifold of even dimensionality $2n$ whose points are described by a system of coördinates x^α ($\alpha, \beta, \gamma, \rho, \sigma, \tau = 1, \cdots, 2n$), and let $a_{\alpha\beta}$ be a nonsingular skewsymmetric covariant tensor whose components are analytic functions of the x's. The manifold equipped with such a tensor is denoted by L_{2n}, and the tensor is called its *fundamental tensor*. It is at once verified that the skewsymmetric quantity

$$K_{\alpha\beta\gamma} = \partial_\alpha a_{\beta\gamma} + \partial_\beta a_{\gamma\alpha} + \partial_\gamma a_{\alpha\beta} \qquad (\partial_\alpha = \partial/\partial x^\alpha) \tag{1.1}$$

is a covariant tensor, which we call the *curvature tensor* of L_{2n}. This is the simplest tensor involving the derivatives of the fundamental tensor. The curvature tensor evidently satisfies the identity

$$\partial_\rho K_{\alpha\beta\gamma} - \partial_\alpha K_{\rho\beta\gamma} - \partial_\beta K_{\alpha\rho\gamma} - \partial_\gamma K_{\alpha\beta\rho} = 0 \tag{1.2}$$

which is a tensorial equation, holding for all coördinate systems.

In terms of Cartan's well known notation in the exterior calculus,[2] there is associated with $a_{\alpha\beta}$ the *exterior differential form* of degree two

$$\Omega = a_{\alpha\beta}[dx^\alpha dx^\beta] \tag{1.3}$$

which we call the *fundamental form* of L_{2n}. As $a_{\alpha\beta}$ is nonsingular, Ω is also called *nonsingular*. The *exterior derivative* of Ω is

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[1] The author wishes to express his thanks to Professor J. M. Thomas for reading the manuscript and giving valuable criticisms.

[2] See e. g. E. Cartan, *Leçons sur les invariants intégraux*, chap. VII.

$$\Omega' = \tfrac{1}{3}K_{\alpha\beta\gamma}[dx^\alpha dx^\beta dx^\gamma] \tag{1.4}$$

and (1.2) is equivalent to the well known fact $\Omega'' = 0$.

An L_{2n} is called *flat* if its curvature tensor vanishes. Evidently, *every L_2 is flat.* For a flat L_{2n} we have $\Omega' = 0$ so that Ω itself is the exterior derivative of a pfaffian form.[3] Taking, by a change of coördinates, this pfaffian in its well known canonical form [4] we find that Ω may be written as

$$[dx^1 dx^{2n}] + [dx^2 dx^{2n-1}] + \cdots,$$

the last term being certainly $[dx^n dx^{n+1}]$ because of the nonsingularity of Ω. Hence in the new coördinate system the fundamental tensor assumes the form

$$a_{\alpha\beta} = \tfrac{1}{2}\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \tag{1.5}$$

where J denotes the matrix of order n with 1's on the secondary diagonal and 0's elsewhere and 0 denotes the zero matrix of order n.

It is also clear that there are other coördinate systems for which the components of $a_{\alpha\beta}$ are any constants. The converse is evident, and we have

THEOREM 1. *An L_{2n} is flat if and only if there exists a coördinate system for which the components of the fundamental tensor are constants.*

2. Conformal spaces. An L_{2n} and an $'L_{2n}$ over the same underlying manifold are said to be *conformal* to each other if their respective fundamental tensors $a_{\alpha\beta}(x)$ and $'a_{\alpha\beta}(x)$, referred to the same coördinate system, are connected by a relation of the form

$$'a_{\alpha\beta} = \phi a_{\alpha\beta}, \tag{2.1}$$

where ϕ is a nonvanishing scalar function of the coördinates in question. Evidently, *every L_2 is conformal to every other L_2.* We shall assume $2n > 2$ throughout this section. By (2.1) we find

$$'K_{\alpha\beta\gamma} = \phi(K_{\alpha\beta\gamma} + a_{\beta\gamma}\partial_\alpha \log\phi + a_{\gamma\alpha}\partial_\beta \log\phi + a_{\alpha\beta}\partial_\gamma \log\phi). \tag{2.2}$$

Since $a_{\alpha\beta}$ is nonsingular, it has an inverse $a^{\alpha\beta}$ (skewsymmetric in α, β) defined by $a^{\alpha\gamma}a_{\gamma\beta} = \delta_\beta{}^\alpha$ where $\delta_\beta{}^\alpha$ is 1 or 0 according as α is or is not equal to β. Define the covariant vector

$$K_\alpha = K_{\alpha\beta\gamma}a^{\beta\gamma}$$

and call it the *curvature vector of L_{2n}.* Since by (2.1) we have $'a^{\alpha\beta} = \phi^{-1}a^{\alpha\beta}$, we find from (2.2)

$$'K_\alpha = K_\alpha - (2n-2)\partial_\alpha \log\phi. \tag{2.3}$$

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[3] See e. g. E. Goursat, *Leçons sur le problème de Pfaff*, p. 105.

[4] Goursat, *loc. cit.*, p. 42.

Consequently the skewsymmetric covariant tensor

$$b_{\alpha\beta} = \partial_\alpha K_\beta - \partial_\beta K_\alpha \tag{2.4}$$

is an obsolute conformal invariant:

$$'b_{\alpha\beta} = b_{\alpha\beta}. \tag{2.5}$$

We call $b_{\alpha\beta}$ the *first conformal curvature tensor* of L_{2n}.

The result of eliminating the derivatives of $\log \phi$ in (2.2) by means of (2.3) may be written

$$'c_{\alpha\beta\gamma} = \phi\, c_{\alpha\beta\gamma} \tag{2.6}$$

where

$$c_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} + (K_\alpha a_{\beta\gamma} + K_\beta a_{\gamma\alpha} + K_\gamma a_{\alpha\beta})/(2n-2). \tag{2.7}$$

Thus the skewsymmetric covariant tensor $c_{\alpha\beta\gamma}$ is a relative conformal invariant which we call the *second conformal curvature tensor* of L_{2n}.

Solving (2.7) for $K_{\alpha\beta\gamma}$ and substituting into (1.2) give

$$a_{\rho\alpha}b_{\beta\gamma} + a_{\rho\beta}b_{\gamma\alpha} + a_{\rho\gamma}b_{\alpha\beta} + b_{\rho\alpha}a_{\beta\gamma} + b_{\rho\beta}a_{\gamma\alpha} + b_{\rho\gamma}a_{\alpha\beta} = c_{\rho\alpha\beta\gamma}, \tag{2.8}$$

where for the sake of brevity $c_{\rho\alpha\beta\gamma}$ indicates the aggregate of terms independent of the derivatives of K_α. Because of (1.1) and (2.7) this expression may be written

$$\begin{aligned} c_{\rho\alpha\beta\gamma} = {} & (2n-2)(\partial_\rho c_{\alpha\beta\gamma} - \partial_\alpha c_{\rho\beta\gamma} - \partial_\beta c_{\rho\gamma\alpha} - \partial_\gamma c_{\rho\alpha\beta}) \\ & + (K_\rho c_{\alpha\beta\gamma} - K_\alpha c_{\rho\beta\gamma} - K_\beta c_{\rho\gamma\alpha} - K_\gamma c_{\rho\alpha\beta}) \end{aligned}$$

(which is a skewsymmetric covariant tensor). Transvecting (2.8) by $a^{\rho\gamma}$ and writing $c_{\alpha\beta} = a^{\rho\gamma}c_{\rho\alpha\beta\gamma}$ we find

$$(2n-4)b_{\alpha\beta} = ba_{\alpha\beta} - c_{\alpha\beta}$$

where $b = a^{\alpha\beta}b_{\alpha\beta}$. Transvecting the above equation by $a^{\alpha\beta}$ and writing $c = a^{\alpha\beta}c_{\alpha\beta}$ we obtain $2(2n-2)b = -c$ so that

$$-2(2n-2)(2n-4)b_{\alpha\beta} = ca_{\alpha\beta} + 2(2n-2)c_{\alpha\beta}. \tag{2.9}$$

We observe that (2.4) and (2.7) immediately imply

THEOREM 2. *The first conformal curvature tensor vanishes if, and only if, the curvature vector is a gradient and the second conformal curvature tensor vanishes if, and only if, the curvature tensor is of the form*

$$K_{\alpha\beta\gamma} = g_\alpha a_{\beta\gamma} + g_\beta a_{\gamma\alpha} + g_\gamma a_{\alpha\beta} \tag{2.10}$$

where g_α is a covariant vector.

When $2n = 4$, (2.10) consists of four equations which are directly verified to be solvable for the four unknowns g_1, g_2, g_3, g_4. Hence

THEOREM 3. *The second conformal curvature tensor of every L_4 vanishes identically.*

An L_{2n} is said to be *conformally flat* if a scalar ϕ exists such that the $'L_{2n}$ defined by (2.1) is flat. Suppose that L_{2n} is conformally flat. If $'L_{2n}$ is the corresponding flat space, then $'K_{\alpha\beta\gamma} = 0$ and therefore $'K_\alpha = 0$, $'b_{\alpha\beta} = 0$, $'c_{\alpha\beta\gamma} = 0$ so that by (2.5) and (2.6) we have $b_{\alpha\beta} = 0$, $c_{\alpha\beta\gamma} = 0$. Suppose conversely that the latter hold. Then K_α is a gradient (by the first part of Theorem 2) and $'c_{\alpha\beta\gamma} = 0$ (by (2.6)). The former fact shows, by (2.3), that we can make $'K_\alpha$ vanish by properly choosing ϕ. From $'c_{\alpha\beta\gamma} = 0$ and $'K_\alpha = 0$ we have $'K_{\alpha\beta\gamma} = 0$ by (2.7), whence $'L_{2n}$ is flat and therefore L_{2n} is conformally flat. Thus we have proved

THEOREM 4. *A necessary and sufficient condition that an L_{2n} $(n > 1)$ be conformally flat is that the first and second conformal curvature tensors vanish.*

When $2n > 4$, the vanishing of the first conformal curvature tensor is a consequence of the vanishing of the second conformal curvature tensor, as is at once seen from (2.9). Hence, in view of Theorem 3 for $2n = 4$, Theorem 4 may be refined into

THEOREM 5. *A necessary and sufficient condition that an L_{2n} be conformally flat is that the first conformal curvature tensor vanish for $2n = 4$ and the second conformal curvature tensor vanish for $2n > 4$.*

Given a conformally flat L_{2n}, the determination of the corresponding flat space is now immediate. For in this case we have, by Theorems 4, 2 equation (2.10) where $g_\alpha = -K_\alpha/(2n-2)$. The vector g_α is a gradient and so may be written in the form

$$g_\alpha = -\partial_\alpha \log \phi, \tag{2.11}$$

where ϕ is different from zero and is determined save for a constant factor. Because of (1.1) and (2.11), (2.10) may be written

$$\partial_\alpha(\phi a_{\beta\gamma}) + \partial_\beta(\phi a_{\gamma\alpha}) + \partial_\gamma(\phi a_{\alpha\beta}) = 0 \tag{2.12}$$

which shows that $\phi a_{\alpha\beta}$ is the required fundamental tensor of the corresponding flat space. (2.12) also implies the existence of a *covariant vector* a_α, determined save for an additive gradient, such that

$$\phi a_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha. \tag{2.13}$$

3. Integrating factors of non-singular exterior differential forms of degree two. The theorems of the preceding section incidentally solve a

particular problem in the exterior calculus. An exterior differential form (of any degree) is called an *exact differential* if its exterior derivative vanishes. An exterior differential form is said to admit an *integrating factor* if it can be made an exact differential by multiplication of a scalar factor.[5] Except for the case when the degree of the form is one (pfaffian form), a necessary and sufficient condition for a form to admit an integrating factor has not yet been found.

It is clear that the form Ω defined by (1.3) *admits an integrating factor if and only if the corresponding* L_{2n} *is conformally flat.* When $2n > 4$, Theorems 5 and 2 imply that L_{2n} is conformally flat if and only if (2.10) holds. Write the latter in the form

$$\Omega' = [\omega\Omega], \tag{3.1}$$

where ω denotes the pfaffian $g_a dx^a$. Thus Ω is a *factor* of Ω',[6] and we have

THEOREM 6. *A necessary and sufficient condition that a nonsingular exterior differential form of degree two in an even number* (> 4) *of variables admit an integrating factor is that the form itself be a factor of its exterior derivative.*

Again by Theorems 5 and 2, an L_4 is conformally flat if and only if K_a is a gradient, i. e. the pfaffian form $\theta = K_a dx^a$ is an exact differential. Now

$$\theta = K_a dx^a = K_{a\beta\gamma} a^{\beta\gamma} dx^a = a^{\alpha\beta} K_{\alpha\beta\gamma} dx^\gamma = a^{\alpha\beta}\theta_{\alpha\beta}$$

where $\theta_{\alpha\beta} = K_{\alpha\beta\gamma} dx^\gamma$ is a system of (six) pfaffian forms, said to be *associated* with the form Ω'.[7] Calling $\theta = a^{\alpha\beta}\theta_{\alpha\beta}$ the *contraction*[8] of this associated system, we have

THEOREM 7. *A necessary and sufficient condition that a nonsingular exterior differential form of degree two in four variables admit an integrating factor is that the pfaffian form, which is the contraction of the system of pfaffian forms associated with the exterior derivative of the given form, be an exact differential.*

4. Infinitesimal automorphisms of L_{2n}. We conclude this paper by giving a brief study of the infinitesimal automorphisms of an L_{2n}, which are, by definition, infinitesimal (point) transformations leaving invariant the fundamental form Ω. In the same way as Killing's equation in Riemannian

[5] Goursat, *loc. cit.*, p. 118.
[6] Goursat, *loc. cit.*, p. 97.
[7] Goursat, *loc. cit.*, p. 126.
[8] This operation is unfamiliar in the exterior calculus.

geometry is obtained, we find that a contravariant vector v^α is the vector of an infinitesimal automorphism of L_{2n} if and only if it satisfies the equation

$$v^\rho \partial_\rho a_{\alpha\beta} + a_{\rho\beta}\partial_\alpha v^\rho + a_{\alpha\rho}\partial_\beta v^\rho = 0. \tag{4.1}$$

Introducing the *covariant associate* $v_\alpha = a_{\alpha\beta}v^\beta$ of v^α we may write this equation in the form

$$\partial_\alpha v_\beta - \partial_\beta v_\alpha = -v_\rho K^\rho{}_{.\alpha\beta}, \tag{4.2}$$

where $K^\rho{}_{.\alpha\beta} = a^{\rho\gamma}K_{\gamma\alpha\beta}$.

We shall only try to solve (4.2) in particular cases. If L_{2n} is flat, we have $K^\rho{}_{.\alpha\beta} = 0$ so that (4.2) implies that a covariant vector is the vector of an infinitesimal automorphism of a flat L_{2n} if and only if it is a gradient. More generally, if L_{2n} is conformally flat we may, using (2.10) and (2.11), write (4.2) in the form

$$\partial_\alpha(\phi v_\beta) - \partial_\beta(\phi v_\alpha) = v\phi a_{\alpha\beta}, \tag{4.3}$$

where $v = v^\rho g_\rho$, which can be shown to be a constant. In fact, if we operate on (4.3) with ∂_γ, perform a cyclic permutation of α, β, γ and then add the three equations thus obtained, we find, on account of (2.12), $a_{\beta\gamma}\partial_\alpha v + a_{\gamma\alpha}\partial_\beta v + a_{\alpha\beta}\partial_\gamma v = 0$, whence $\partial_\alpha v = 0$ and $v = \text{const.}$ Consequently, in view of (2.13), the general solution of (4.3) is of the form

$$\phi v_\alpha = v a_\alpha + \partial_\alpha w. \tag{4.4}$$

If we introduce the *contravariant associate* $g^\alpha = a^{\alpha\beta}g_\beta$ of g_α and transvect (4.4) by g^α, we find

$$g^\alpha \partial_\alpha w = -v(\phi + a) \tag{4.5}$$

where $a = g^\alpha a_\alpha$. Hence

THEOREM 8. *A covariant vector v_α is the vector of an infinitesimal automorphism of a conformally flat L_{2n} if and only if it is given by* (4.4) *where v is a constant and w is a solution of the linear equation* (4.5).

In this conformally flat case ($n > 1$) it is easily seen that $v = 0$ if and only if v_α satisfies

$$v_\alpha(\partial_\beta v_\gamma - \partial_\gamma v_\beta) + v_\beta(\partial_\gamma v_\alpha - \partial_\alpha v_\gamma) + v_\gamma(\partial_\alpha v_\beta - \partial_\beta v_\alpha) = 0,$$

i.e., v_α is proportional to a gradient $\partial_\alpha \rho$, and this is the case if and only if the trajectories of the 1-parameter group of automorphisms generated by v^α are at the same time trajectories of other 1-parameter groups of automorphisms, the generators of the latter being then necessarily v^α multiplied by functions of ρ.

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OSCULATING CONICS OF THE PLANE SECTIONS THROUGH A POINT OF A SURFACE.*

By Buchin Su.

1. Introduction. According to the classical theorem of Moutard all the osculating conics of ∞^2 plane sections through a generic point O of an analytic surface immersed in a three-dimensional projective space are distributed on ∞^1 Moutard quadrics each of which corresponds to a non-asymptotic tangent of the surface at O. In connection with the definition of Darboux pencils of quadrics we have studied [1] Moutard pencils of quadrics (so called because each of these pencils contains a Moutard quadric). This notion has been generalized ingeniously by Louis Green [2] who shows, among other things, that two arbitrary non-asymptotic tangents at O suffice to determine a certain pencil of quadrics having a contact of the second order with the surface at O.

In studying the residual osculating conic on a given Moutard quadric Chenkuo Pa has recently proved that [3] if a quadric through the two asymptotic tangents at O possesses one and only one osculating conic of the surface at the same point, then it must necessarily belong to a Moutard pencil. Combining the investigation of L. Green with that of Pa we are naturally led to inquire whether the osculating conics belonging to *different* Moutard quadrics at O may be regulated in such a manner that they lie on a quadric of certain pencils.

The purpose of this paper is to solve the problem of determining a quadric by means of an osculating conic of a given direction and another osculating conic of another given direction with a suitably chosen plane, showing that on the quadric thus obtained there exists a third osculating conic. Thus all the osculating conics of the plane sections through O are newly distributed, three by three, on a quadric having a contact of the second order with the surface at O and meeting it in a curve containing these tangents as two of its triple tangents.

2. Analytic basis. If the four homogeneous projective coördinates x of a variable point on an analytic non-ruled surface S in ordinary space are given

* Received July 17, 1941.

[1] Buchin Su and Ichida, "On certain cones connected with a surface in the affine space," *Japanese Journal of Mathematics*, vol. 10 (1933), pp. 209-216.

[2] Louis Green, "Systems of quadrics associated with a point of a surface," *American Journal of Mathematics*, vol. 60 (1938), pp. 649-666.

[3] C. Pa, "On the quadrics of Moutard," not yet published.

as analytic functions of two independent variables u, v, and if the parametric net on S is the asymptotic net, then the functions x are solutions of a completely integrable system of differential equations in Fubini's canonical form

$$(1) \qquad \begin{cases} x_{uu} = \theta_u x_u + \beta x_v + px, \\ x_{vv} = \gamma x_u + \theta_v x_v + qx \end{cases} \qquad (\theta = \log \beta\gamma).$$

The coefficients of these equations are functions of u, v and satisfy certain conditions of integrability.

The coördinates of a point O' on S near O can be represented by the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$. For the non-homogeneous coördinates x, y, z of O' defined by putting

$$(2) \qquad x = x_2/x_1, \qquad y = x_3/x_1, \qquad z = x_4/x_1$$

we have the canonical expansion in the form

$$(3) \qquad z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3) + \phi_4(x, y) + \cdots,$$

where

$$(4) \qquad \phi_4(x, y) = \frac{1}{12}\{\beta\phi x^4 - 4\beta\psi x^3 y - 6\theta_{uv} x^2 y^2 - 4\gamma\phi x y^3 + \gamma\psi y^4\}$$

and

$$(5) \qquad \phi = \frac{\partial \log \beta\gamma^2}{\partial u}, \qquad \psi = \frac{\partial \log \beta^2\gamma}{\partial v}.$$ [4]

Consider the section of the surface produced by a plane through a non-asymptotic tangent t_n whose equations are

$$(6) \qquad z = 0, \qquad y - nx = 0;$$

the equation of this plane is evidently of the form

$$(7) \qquad z = \rho(y - nx),$$

so that the osculating conic of the plane section at O is given by the equations (7) and

$$(8) \qquad \begin{aligned} -x^2 + Y[\{(\rho/3n^2)(\beta + \gamma n^3) - (1/n)\}x \\ - (\rho^2/n^2)\{(1/3n\rho)(\beta - 2\gamma n^3) + (1/n)\phi_4(1, n) \\ - (1/9n^2)(\beta + \gamma n^3)^2\}Y + (\rho/n)] = 0, \end{aligned}$$

where we have placed

$$(9) \qquad Y = y - nx.$$

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[4] Cfr. E. P. Lane, "Power series expansions in the neighborhood of a point on a surface," *Proceedings of the National Academy of Sciences*, vol. 13 (1927), pp. 808-813.

Elimination of ρ from (7) and (8) gives the equation of the Moutard quadric belonging to the given tangent t_n, namely,

$$(10)\qquad \begin{aligned} &36n^3(z-xy)+12n^2(2\beta-\gamma n^3)xz-12n(\beta-2\gamma n^3)yz \\ &\quad+\{4(\beta+\gamma n^3)^2-36n\phi_4(1,n)\}z^2=0. \end{aligned}$$

3. Quadrics having a contact of the second order with a surface. The equation of the most general quadric through the two asymptotic tangents of the surface S at the point O is

$$(11)\qquad xy+k_1z+k_2xz+k_3yz+k_4z^2=0,$$

where k_1, k_2, k_3, k_4 are arbitrary functions of u and v. The conic intersection of this quadric and the plane (7) is expressed by (7) and

$$(12)\qquad \begin{aligned} -x^2+Y[\{-(\rho/n)(k_2+k_3n)-1/n\}x \\ -(\rho^2/n)((1/\rho)k_3+k_4)Y-k_1(\rho/n)]=0. \end{aligned}$$

In order that this conic be the osculating conic (8) of the plane section of the surface produced by the plane (7), it is necessary and sufficient that $k_2=-1$ and

$$(13)\qquad \begin{cases} k_2+k_3n=-(1/3n)(\beta+\gamma n^3), \\ k_3-(1/3n^2)(\beta-2\gamma n^3)=\rho\{(1/n^2)\phi_4(1,n) \\ \qquad\qquad -(1/9n^3)(\beta+\gamma n^3)^2-k_4\}. \end{cases}$$

Suppose that two arbitrary non-asymptotic tangents to S at O are given, respectively, by the equations

$$(14)\qquad z=0,\qquad y-n_ix=0 \qquad (i=1,2)$$

and that through each of them there passes a plane

$$(15)\qquad z=\rho_i(y-n_ix) \qquad (i=1,2)$$

on which lies the corresponding osculating conic K_i of the section at O. If the quadric (11) passes through both conics K_1 and K_2 the conditions (13) must hold for $n=n_i$, $\rho=\rho_i$ $(i=1,2)$ and the same set of k_2, k_3 and k_4; thus

$$(16)\qquad \begin{cases} k_2+k_3n_1=-(1/3n_1)(\beta+\gamma n_1^3), \\ k_2+k_3n_2=-(1/3n_2)(\beta+\gamma n_2^3); \end{cases}$$

$$(17)\qquad \begin{cases} k_3-(1/3n_1^2)(\beta-2\gamma n_1^3) \\ \qquad=\rho_1\{(1/n_1^2)\phi_4(1,n_1)-(1/9n_1^3)(\beta+\gamma n_1^2)^2-k_4\}, \\ k_3-(1/3n_2^2)(\beta-2\gamma n_2^3) \\ \qquad=\rho_2\{(1/n_2^2)\phi_4(1,n_2)-(1/9n_2^3)(\beta+\gamma n_2^3)^2-k_4\}. \end{cases}$$

Solving (16) for k_2 and k_3 we obtain

$$(18)\qquad \begin{cases} k_2 = -(1/3)\{(\beta/n_1n_2)(n_1+n_2) - \gamma n_1n_2\}, \\ k_3 = (1/3)\{\beta/n_1n_2 - \gamma(n_1+n_2)\}. \end{cases}$$

The plane through one of the tangents t_1 and t_2 may be assigned arbitrarily and that through the other tangent is perfectly determined, since (17) gives k_4 for a given ρ_1 and therefore determines ρ_2, except when the quadric thus obtained coincides with a non-Moutard quadric of the Moutard pencil belonging to the tangent t_1 or t_2. In the exceptional case the plane (14) coincides with the tangent plane of the surface at the point O, so that no osculating conic can be obtained. That is to say, there is one and only one osculating conic on a non-Moutard quadric of any Moutard pencil, as was shown by Pa.

Let us consider the general case, confining ourselves to the determination of the other osculating conics on the quadric in consideration. Writing the equation of the plane of such a conic, if any, in the form

$$(19)\qquad z = \rho_3(y - n_3x),$$

we have

$$(20)\qquad \begin{cases} k_2 + k_3n_3 = -(1/3n_3)(\beta + \gamma n_3^3), \\ k_3 - (1/3n_3^2)(\beta - 2\gamma n_3^2) \\ \qquad = \rho_3\{(1/n_3^2)\phi_4(1, n_3) - (1/9n_3)(\beta + \gamma n_3^3)^2 - k_4\}. \end{cases}$$

The last condition evidently determines ρ_3 and consequently the plane (19) provided that the quadric is not the Moutard quadric belonging to the tangent t_3. Substituting the values of k_2 and k_3 given by (18) into the first equation of (20) and reducing, we are led to the relation

$$(21)\qquad n_1n_2n_3 = -\beta/\gamma,$$

which shows that the third tangent t_3 is completely determinate when the first two t_1 and t_2 are known.

By virtue of (21) we can express (18) in the symmetrical form

$$(22)\qquad \begin{cases} k_2 = \tfrac{1}{3}\gamma(n_1n_2 + n_2n_3 + n_3n_1), \\ k_3 = -\tfrac{1}{3}\gamma(n_1 + n_2 + n_3). \end{cases}$$

Thus we arrive at the following conclusion:

Given two arbitrary non-asymptotic tangents t_1, t_2 to S at a generic point O and one plane π_1 through t_1 we can always determine a quadric Q through the two asymptotic tangents of S at O such that the osculating conic of the section of S produced by π_1 as well as that of the section produced by

the corresponding plane π_2 through t_2 lies on Q. This quadric, having a contact of the second order with S at O, contains a third osculating conic at O and forms a pencil when the plane π_1 turns about the tangent t_1.

4. A covariant line associated with two non-asymptotic tangents. Let us consider now the correspondence between the planes π_1 and π_2 passing through the non-asymptotic tangents t_1 and t_2, respectively, such that the osculating conics of the sections produced by π_1, π_2 and the asymptotic tangents lie on the same quadric. By virtue of (17) it is easily seen that the planes (15) for $i = 1, 2$ are in the correspondence given by the equation

$$(23)\quad (1/\rho_1)\{k_3 - (1/3n_1^2)(\beta - 2\gamma n_1^3)\} - (1/\rho_2)\{k_3 - (1/3n_2^2)(\beta - 2\gamma n_2^3)\} = (1/n_1^2)\phi_4(1, n_1) - (1/n_2^2)\phi_4(1, n_2) - (1/9n_1^3)(\beta + \gamma n_1^3)^2 + (1/9n_2^3)(\beta + \gamma n_2^3)^2,$$

and are, therefore, in perspective position.

In order to derive the plane, denoted by π_{12}, on which lie all the lines of intersection of the corresponding pairs of planes π_1 and π_2, we have to substitute

$$1/\rho_i = (1/z)(y - n_i x) \qquad (i = 1, 2)$$

into (23) and then to reduce the resulting equation by means of the relations (21) and (22). A simple calculation suffices to obtain the equation of the plane π_{12}, namely,

$$(24)\quad \beta(2n_3 - p_1)x + \gamma n_3(p_2 - 3n_1n_2)y = [\tfrac{1}{4}\{-\frac{\beta^2\phi}{\gamma}\frac{n_1 + n_2}{n_1^2n_2^2} + 4\frac{\beta^2\psi}{\gamma}\frac{1}{n_1n_2} - 4\beta\phi + \beta\psi(n_1 + n_2)\} + \tfrac{1}{3}\{\frac{\beta^3}{\gamma}\frac{n_1^2 + n_1n_2 + n_2^2}{n_1^3n_2^3} - \beta\gamma(n_1^2 + n_1n_2 + n_2^2)\}]z,$$

where we have placed

$$(26)\qquad p_1 = n_1 + n_2 + n_3, \qquad p_2 = n_1n_2 + n_2n_3 + n_3n_1.$$

If we take the three tangents t_i $(i = 1, 2, 3)$ at O given by

$$(27)\qquad z = y - n_i x = 0 \qquad (i = 1, 2, 3),$$

where n_1, n_2, n_3 satisfy (21), then there are three planes π_{12}, π_{23}, π_{31} which correspond to the pairs of tangents t_1, t_2; t_2, t_3 and t_3, t_1 respectively. These planes are evidently concurrent in a line l associated with the tangents t_1 and t_2 (and consequently t_3) of the surface S at O.

From the equation (24), and that obtained by interchanging n_1 and n_2, the equations of l follow immediately. The result of carrying out the computation is as follows:

$$(28)\quad x:y:z = \tfrac{1}{4}(-15\beta\psi + 2\gamma p_2\phi - \gamma p_1 p_2\psi) - \gamma^2 p_2^{\,2} + \tfrac{1}{3}\gamma^2 p_1^{\,2} p_2$$
$$:\tfrac{1}{4}(-15\beta\phi - 2\beta p_1\psi - \gamma p_1 p_2\phi) - \beta\gamma p_1^{\,2} - \tfrac{1}{3}\gamma^2 p_1 p_2^{\,2} : 9\beta + \gamma p_1$$

Several particular cases of the above result are worth noting. If the three tangents are those of Darboux, then $p_1 = p_2 = 0$ and, consequently, the corresponding line l is the canonical line of the first kind for which $k = -5/12$, which we shall designate as *the first axis of Bompiani.*[5]

If two of the tangents lie in distinct Segre directions, the third is seen to lie in that Darboux direction which is conjugate to the residual Segre direction. Every Darboux direction thus determines a covariant line l. Putting

$$n_1 = (\beta/\gamma)^{1/3}\omega^{r+1}, \qquad n_2 = (\beta/\gamma)^{1/3}\omega^{r+2}, \qquad n_3 = (\beta/\gamma)^{1/3}\omega^{r},$$

where $\omega^3 = 1$, $\omega \neq 1$ and $r = 1, 2, 3$, so that

$$p_1 = -2(\beta/\gamma)^{1/3}\omega^r, \qquad p_2 = 2(\beta/\gamma)^{2/3}\omega^{2r},$$

we obtain the covariant line l_r

$$x:y:z = -(11/20)\psi - (4/15)\beta^{1/3}\gamma^{2/3}\omega^r + (1/5)(\gamma/\beta)^{1/3}\phi\omega^{2r}$$
$$: -(11/20)\phi + (1/5)(\beta/\gamma)^{1/3}\omega^r - (4/15)\beta^{2/3}\gamma^{1/3}\omega^{2r} : 1.$$

The harmonic polar of the tangent plane $z = 0$ with respect to the trihedron formed by l_1, l_2, l_3 is easily found to be *the canonical line of the first kind for which* $k = -11/20$.

If two of the tangents are conjugate, namely, such that their directions are n, $-n$, the third must lie in the direction $\beta/\gamma n^2$, so that

$$p_1 = \beta/\gamma n^2, \qquad p_2 = -n^2$$

and the covariant line l_n corresponding to any non-asymptotic direction n (and its conjugate $-n$) is given by the equations

$$(29)\quad x:y:z = \{-(7/16)\psi - (\gamma^2/8\beta)n^4 - (1/16)(\gamma/\beta)\phi n^2 - (1/24)\beta(1$$
$$:\{-(7/16)\phi - (\beta^2/8\gamma)(1/n^4) - (1/16)(\beta/\gamma)\psi(1/n^2) - (1/24)\gamma^2$$

[5] Cf. L. Green, *loc. cit.*[2], p. 654.

In particular, for the Darboux directions

$$n = -(\beta/\gamma)^{1/3}\omega^r \qquad (r = 1, 2, 3)$$

(and the Segre directions) we have the lines:

$$x : y : z = \{-(7/16)\psi - (1/6)\beta^{1/3}\gamma^{2/3}\omega^r - (1/16)(\gamma/\beta)^{1/3}\omega^{2r}\}$$
$$: \{-(7/16)\phi - (1/16)(\beta/\gamma)^{1/3}\omega^r - (1/6)(\beta^{2/3}\gamma^{1/3}\omega^{2r}\} : 1,$$

where $r = 1, 2, 3$. In consequence, the harmonic polar of the tangent plane $z = 0$ with respect to the trihedron formed the these lines is *the canonical line of the first kind for which* $k = -7/16$.

If two of the tangents coincide in a non-asymptotic direction n, the third must lie in the direction $-\beta/\gamma n^2$, so that

$$p_1 = -(1/\gamma n^2)(\beta - 2\gamma n^3), \qquad p_2 = -(1/\gamma n)(2\beta - \gamma n^3).$$

By means of these values there is no difficulty in determining the line (28) in question. Without expressing the equations explicitly we shall only consider two interesting cases, namely, the lines corresponding to a Segre direction and a Darboux direction. In the latter case we have

$$n = -(\beta/\gamma)^{1/3}\omega^r.$$

so that

$$p_1 = -3(\beta/\gamma)^{1/3}\omega^r, \qquad p_2 = -3(\beta/\gamma)^{2/3}\omega^{2r}$$

and the line becomes

$$x : y : z = \{(1/3)\psi - (1/12)(\gamma/\beta)^{1/3}\phi\omega^{2r} - \gamma(\beta/\gamma)^{1/3}\omega^r\}$$
$$: \{-(1/3)\phi + (1/12)(\beta/\gamma)^{1/3}\psi\omega^r\} : 1.$$

These form a trihedron with respect to which the harmonic polar of the tangent plane $z = 0$ is obviously *the first axis of Čech*.

In the similar way we obtain a covariant line corresponding to every tangent of Segre; the harmonic polar in question is *the canonical line of the first kind for which* $k = -7/16$.

5. The quadrics of Darboux. Let us confine ourselves to an important case which can be derived from the general result stated in Section **3**, namely, the case where the tangents are in Darboux directions. The quadric in question must be a Darboux quadric, since (22) gives $k_2 = k_3 = 0$ for the roots n_1, n_2, n_3 of the equation

$$\beta + \gamma n^3 = 0,$$

so that the equation of the quadric becomes

$$xy - z + k_4 z^2 = 0. \tag{30}$$

The planes of osculating conics lying on this Darboux quadric are

$$z = \rho_i (y - n_i x) \qquad (i = 1, 2, 3), \tag{31}$$

where

$$1/\rho_i = -(1/\beta)[\phi_4(1, n_i) - n_i^2 k_4] \tag{32}$$

and

$$n_i = -(\beta/\gamma)^{\frac{1}{3}} \omega^i \qquad (i = 1, 2, 3) \tag{33}$$

In order that these planes be concurrent in a line it is necessary and sufficient that

$$\begin{vmatrix} n_1 & 1/\rho_1 & 1 \\ n_2 & 1/\rho_2 & 1 \\ n_3 & 1/\rho_3 & 1 \end{vmatrix} = 0$$

or

$$k_4 = -\tfrac{1}{2}\theta_{uv}. \tag{34}$$

Hence we have the following theorem:

Every plane through one tangent of Darboux uniquely determines a Darboux quadric Q such that the three osculating conics of sections produced by this plane and two others each passing through the remaining tangents of Darboux lie on Q. The planes of these conics are concurrent if, and only if, Q is the quadric of Wilczynski-Bompiani.

The common line of these planes, in the last case, can be found by substituting (34), (32) into (31), so that the equation of each plane is

$$n_i[x + (5/12)\psi z] - [y + (5/12)\phi z] = 0.$$

Hence *the planes of the three osculating conics on the quadric of Wilczynski-Bompiani are concurrent in the first axis of Bompiani.*

Since the expression (32) of $1/\rho_i$ for a given n_i is linear in k_4, we may conclude that if π_0, π_L, π_W be the tangent plane and the planes of osculating conics of sections through a non-asymptotic tangent and lying, respectively, on the quadric of Lie and that of Wilczynski-Bompiani and if π denotes the plane through the same tangent such that the double ratio

$$(\pi_0 \pi_L, \pi \pi_W) = h,$$

the quadric determined by π is the Darboux quadric

$$xy - z - \{\tfrac{1}{2}\theta_{uv} + (1-h)\beta\gamma\}z^2 = 0. \tag{35}$$

Moreover, by means of the pencil of Darboux quadrics, we can also determine the first and the second axes of Bompiani. In fact, the three planes of osculating conics lying on the quadric (30) form a trihedron of edges

$$n_i x - y + (1/\rho_i)z = 0, \qquad n_j x - y + (1/\rho_j)z = 0,$$

where $i \neq j$, $i, j = 1, 2, 3$ and ρ_i is given by (32). An easy calculation, which we shall omit here, gives the equations of the harmonic polar of the tangent plane with respect to the trihedron, namely,

$$x : y : z = -(5/12)\psi : -(5/12)\phi : 1,$$

which represent the first axis of Bompiani.

In the similar way the three conics in consideration intersect each other at three points distinct from O; when the quadric varies in the pencil of Darboux the plane of these points of intersection describes a flat pencil whose axis is given by

$$z = 0, \qquad (5/12)\phi x + (5/12)\psi y + 1 = 0,$$

that is, *the second axis of Bompiani.*

6. The curve of intersection of a surface and a quadric. As was shown in Section **3**, the planes

$$z = \rho_i(y - n_i x) \qquad (i = 1, 2, 3)$$

determine uniquely a quadric whose equation is

$$xy - z + k_2 xz + k_3 zy + k_4 z^2 = 0, \tag{36}$$

provided that the directions n_1, n_2, n_3 are connected by the relation (21). From (36) it follows that the quadric thus determined must have a contact of the second order with the surface S at O.

By means of the relation (22) we can easily demonstrate that the triple tangents of the curve of intersection of (36) and S at O coincide with the three tangents

$$(y - n_1 x)(y - n_2 x)(y - n_3 x) = 0. \tag{37}$$

Hence *the quadrics corresponding to the planes through a non-asymptotic tangent t_1 or t_2 form a pencil of quadrics having a contact of the second order with S at O and meeting S in a one-parameter family of curves each containing*

these tangents as two of its triple-point tangents. The residual tangent t_3 at O is the same for all the curves of the family and is the common tangent of all the osculating conics each of which lies on a quadric of the pencil.

Conversely, suppose that a quadric Q has a contact of the second order with the surface S at O; the curve of intersection has O for its triple point. From (3) and (36) we have the equations of the triple-point tangents

$$z = y - n_i x = 0 \qquad (i = 1, 2, 3),$$

where n_1, n_2, n_3 are the roots of the equation

$$\tfrac{1}{3}\beta + k_2 n + k_3 n^2 + \tfrac{1}{3}\gamma n^3 = 0,$$

and therefore

$$n_1 n_2 n_3 = -\beta/\gamma, \quad k_2 = \tfrac{1}{3}\gamma(n_1 n_2 + n_2 n_3 + n_3 n_1), \quad k_3 = -\tfrac{1}{3}\gamma(n_1 + n_2 + n_3).$$

In order to obtain the osculating plane of each branch at O we have to assume that the expansions for the branch which touches the tangent t_i,

$$z = y - n_i x = 0,$$

are given by the equations

$$y = n_i x + Ax^2 + \cdots, \qquad z = nx^2 + \cdots. \tag{38}$$

and then to substitute (38) into the equation

$$\tfrac{1}{3}(\beta x^3 + \gamma y^3) + \phi_4(x, y) + k_2 xy + k_3 xz + k_4 z^2 = 0;$$

the left-hand side should be satisfied identically up to and including the term in x^4. Thus the coefficient A in (38) is given by

$$(1/n_i^2)\phi_4(1, n_i) - k_4 - (1/9n_i^3)(\beta + \gamma n_i^3)^2 = (A/n_i)[k_3 - (1/3n_i^2)(\beta - 2\gamma n_i^3)]. \tag{39}$$

Observing that the osculating plane of the branch (38) at O is

$$Az = n_i(y - n_i x) \tag{40}$$

and comparing (39) with (17), we are led to the following result:

On every quadric Q having a contact of the second order with the surface S at O there are, in general, three osculating conics of the sections of S at O and their planes coincide with the osculating planes to three branches of the curve of intersection of S and Q.

From this we infer that the characterization of the quadric of Wilczynski-

Bompiani stated in Section **3** coincides with the one obtained by L. Green.[6] Similarly, we can put the interesting result of L. Green into a new form containing the osculating conics of sections of the surface.

Several exceptions to the last theorem shall be mentioned.

(i) If the quadric Q determined by an osculating conic C coincides with the Moutard quadric belonging to the tangent of C at O, then one of the two other osculating conics must necessarily coincide with C and the remaining one is *the residual conic of* Q.

(ii) If the quadric Q is a non-Moutard quadric in a Moutard pencil, then the curve of intersection of Q with S at O has two coincident triple-point tangents, the osculating plane to each being the tangent plane of S at O. Therefore there is *only one* osculating conic on a non-Moutard quadric in a Moutard pencil as Pa has shown.

(iii) If the quadric Q is the Moutard quadric belonging to one of the Darboux tangents at O, there is *no residual osculating conic.*

(iv) If Q is a non-Moutard quadric in the Moutard pencil belonging to one of Darboux tangents, it contains *no osculating conic of the surface at* O.

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[6] L. Green, *loc. cit.*[2], p. 654.

THE ASYMPTOTIC BEHAVIOR OF A CLASS OF ENTIRE FUNCTIONS.*

By C. V. NEWSOM.

Introduction. This paper considers the behavior of the function,

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} g(n) z^{sn}, \qquad g(n) = 1/\prod_{j=1}^{s} \Gamma(n+\rho_j),$$

for large values of the complex variable z, if s is an integer > 1, and ρ_j is a complex constant. In particular, if $k \geqq [s/2]+1$ is chosen as an odd integer, it is shown that

$$(2) \quad f(z) = (2\pi)^{(1-s)/2} s^{-\frac{1}{2}} \sum_{p=-(k-1)/2}^{(k-1)/2} e^{sz_p} z_p^{(s+1)/2-\rho} \left[\sum_{j=0}^{n} c_j/(sz_p)^j + \omega(z_p, n)/(sz_p)^n \right]$$

for all values of z satisfying the condition $-\pi/s < \arg z < \pi/s$; wherein $c_0 = 1$; $|z| = |z_p|$; $\arg z_p = \arg z - 2p\pi/s$; $\rho = \sum_{m=1}^{s} \rho_m$; and $\lim_{|z|\to\infty} \omega(z_p, n) = 0$. If k is even the same conclusion results except that the limits upon the first summation are $-k/2$ and $(k-2)/2$ when $-2\pi/s < \arg z < 0$, and are $-(k-2)/2$ and $k/2$ when $0 < \arg z < 2\pi/s$.

The asymptotic behavior of the same function was considered by Van Engen [1] in his dissertation, but the method employed here is sufficiently general that it appears to offer an approach to the study of a large class of entire functions related to (1).

1. A preliminary lemma. The present study employs an important lemma which is essentially a modification of a previous theorem due to the author.[2] The earlier theorem and the present lemma are both concerned with entire functions of the general type (1) in which the coefficient $g(n)$ is such that the function $g(w)$ of the complex variable $w = x + iy$ satisfies the following two conditions:

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* Received January 11, 1942.

[1] Henry Van Engen, "On the asymptotic behavior of analytic functions," *Dissertation*, University of Michigan, 1934.

[2] C. V. Newsom, "On the character of certain entire functions in distant portions of the plane," *American Journal of Mathematics*, vol. 60 (1938), pp. 561-572. Note, also, C. V. Newsom and A. Franck, "Upon the asymptotic representation of functions of the Bessel type," *Bol. Mat.*, vol. 13 (1940), pp. 11-14.

(3) $g(w)$ is single valued and analytic throughout the finite w-plane, and

(4) $|g(w)| < K e^{k\pi y}$, where K is a constant independent of x and y, and k is a positive integer.

The theorem then states that

$$(5)\quad f(z)=\int_{-c-1/2}^{\infty}\{g(x)[\pm z^s]^x(\sin k\pi x)/(\sin \pi x)\}dx-\sum_{m=-c}^{-1} g(m)z^{sm}+\xi_k(c,z),$$

for all values of z satisfying the condition $-\pi < \arg(\pm z^s) < \pi$; wherein c is any arbitrary positive integer; k is the integer appearing in condition (4) above; the upper or the lower of the signs $\pm$ is to be taken according as k is odd or even; and $\lim_{|z|\to\infty} z^c\xi_k(c,z)=0$, irrespective of the value chosen for c.

To inaugurate the study of the representation (5), let k be odd. In view of the fact that $(\sin k\pi x)/(\sin \pi x)=1+2\sum_{p=1}^{(k-1)/2}\cos 2p\pi x$, it follows that the integral of (5) may be rewritten in the form,

$$(6)\quad \int_{-c-1/2}^{\infty}\{g(x)[z^{sx}+2\sum_{p=1}^{(k-1)/2} z_p^{sx}e^{2p\pi ix}\cos 2p\pi x]\}dx,$$

wherein $-\pi/s < \arg z < \pi/s$; $\arg z_p = \arg z - 2p\pi/s$; and $|z| = |z_p|$.

Since $2e^{2p\pi ix}\cos 2p\pi x = 1 + e^{4p\pi ix}$, the expression within the bracket of the integrand of (6) may be considerably simplified. In fact, under the conditions upon $g(w)$ previously given, we have the conclusion,

$$(7)\quad f(z)=\int_{-c-1/2}^{\infty}[\sum_{p=-(k-1)/2}^{(k-1)/2} g(x)z_p^{sx}]dx-\sum_{m=-c}^{-1} g(m)z^{sm}+\xi_k(c,z),$$

in which $-\pi/s < \arg z < \pi/s$; $\arg z_p = \arg z - 2p\pi/s$; and $|z| = |z_p|$.

If k is even, a similar analysis is possible. The conclusion obtained differs from (7), however, in the fact that the limits upon the summation are $-k/2$ and $(k-2)/2$ when $-2\pi/s < \arg z < 0$, and are $-(k-2)/2$ and $k/2$ when $0 < \arg z < 2\pi/s$.

2. Proof of the theorem. To employ the result (7) in the study of the particular problem of this paper, it is merely necessary to specialize $g(x)$

as indicated in (1). As a consequence of a lemma demonstrated by Ford,[3] condition (4) is satisfied when $k \geqq [s/2] + 1$. Thus, it is apparent that a study should now be undertaken of integrals of the type,

$$(8) \qquad \int_{-c-1/2}^{\infty} [z^{sx} / \prod_{j=1}^{s} \Gamma(x + \rho_j)]dx,$$

subject to the restrictions upon z listed after (7).

If $x = (2y - 2\rho + s + 1)/2s$, where $\rho = \sum_{m=1}^{s} \rho_m$, the integral under consideration becomes

$$(9) \qquad z^{(s+1)/2-\rho}/s \int_d^{\infty} [F(y)(sz)^y/\Gamma(y+1)]dy,$$

in which, $d = \rho - sc - s - 1/2$, and $F(y)$ is the ratio of the functions, $\Gamma(y+1)$ and $s^y \prod_{j=1}^{s} \Gamma(y/s + 1/2 + 1/2s - \rho/s + \rho_j)$. The path of integration extends from the lower limit of the integral to positive infinity in the direction of the real y-axis.

The analysis of $\log F(y)$ follows immediately when it is recalled that

$$(10) \qquad \log \Gamma(y) = \tfrac{1}{2} \log 2\pi + (y - \tfrac{1}{2}) \log y - y + \omega(y),$$

in which

$$\omega(y) = B_1/1 \cdot 2y - B_2/3 \cdot 4y^3 + \cdots + (-1)^{n-1}[B_n + \epsilon(y,n)]/(2n-1)(2n)y^{2n-1}$$

and where $\lim_{|y|\to\infty} \epsilon(y,n) = 0$, and $|\arg y| < \pi$. So, for sufficiently large values of y taken upon the path of integration, and after considerable algebraic simplification, it is determined that

$$(11) \qquad \log F(y) = \tfrac{1}{2}(1-s)\log 2\pi + \tfrac{1}{2}\log s + a_1/y + a_2/y^2 + \cdots + [a_n + \alpha(y,n)]/y^n$$

in which the a's are complex constants, and $\lim_{|y|\to\infty} \alpha(y,n) = 0$. Of course, then,

$$(12) \qquad F(y) = s^{\frac{1}{2}} 2\pi^{(1-s)/2} \exp\{a_1/y + a_2/y^2 + \cdots + [a_n + \alpha(y,n)]/y^n\}.$$

As a consequence of a well known principle in the theory of asymptotic series,[4] the exponential factor occurring in this equality may be represented asymptotically by a power series in $1/y$, in which the constant term is unity.

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[3] W. B. Ford, *Asymptotic Developments*, Scientific Series, University of Michigan, XI (1936), p. 61.

[4] Ford, *op. cit.*, pp. 3-4.

This series, in turn, may be reformulated into a factorial series so that integral (9) now becomes:

$$(13)\qquad (2\pi)^{(1-s)/2}s^{-\frac{1}{2}}z^{(s+1)/2-\rho}\{\int_{il}^{\infty}\sum_{j=0}^{n}[c_j(sz)^y/\Gamma(y+j+1)]dy + \int_{il}^{\infty}[\beta(y,n)(sz)^y/\Gamma(y+n+1)]dy\};$$

wherein $c_0 = 1$, and the remaining c's are complex constants; and $\lim_{|y|\to\infty}\beta(y,n) = 0$. In fact, each integral obtained from relation (7) will be of this type.

Integrals similar to those appearing in (13) were studied by Ford [5] in his consideration of functions of exponential type. In his study, however, it was necessary to make the restriction, $|\arg z| < \pi/2$. This condition is satisfied in the present analysis by all the z's except possibly $z_{(k-1)/2}$ and $z_{-(k-1)/2}$ when k is odd, and $z_{(k-2)/2}$ and $z_{-k/2}$ when k is even. Thus with the exception of these particular z's, typified temporarily by the variable u, the part of (13) inclosed within the brace may be written immediately in the following form as the result of the study of Ford:

$$(14)\qquad e^{sz}[\sum_{j=0}^{n}c_j/(sz)^j + \omega(z,n)/(sz)^n];$$

wherein $\lim_{|z|\to\infty}\omega(z,n) = 0$.

For the case when the real part of $u \leqq 0$, a slightly different analysis of (13) is necessary. Referring again to the study of Ford, it will be observed that the first integral of (13) gives a result similar to (14) except that the remainder term does not involve the exponential factor, and is, in fact, in the form $\sum_{j=0}^{n}\alpha_j(u,n)/(su)^j$, where $\lim_{|u|\to\infty}\alpha_j(u,n) = 0$. Thus, for any z having the property that $|\arg z| < \pi/2$, it follows that the product of this remainder term and the factor $e^{-sz}(sz)^m$ produces a function which has the limit zero as $|z| = |u| \to \infty$, when m is any positive integer. Consequently it may be combined with any $\omega(z,n)$ occurring in (14). A similar method serves to dispose of that part of the right member of equality (7) which is outside the integral.

The second integral of (13), when the complex variable u is $z_{(k-1)/2}$, may be written thus:

[5] *Op. cit.*, pp. 65-67.

$$(15) \qquad 1/(su)^n \int_{d+n+1}^{\infty} [\beta(x-n-1,n)(su)^{x-1}/\Gamma(x)]dx$$

$$= 1/(sz_0)^n \{(z_0/u)^n \int_{d+n+1}^{\infty} [P(x)(sz_0)^{x-1}/\Gamma(x)]dx\},$$

in which $P(x) = \beta(x-n-1,n)e^{(x-1)(1-k)\pi i/s}$. Of course, a similar analysis is possible for any other value of u.

As the result of the application of another theorem given by Ford,[6] the function inclosed within the brace of (15), when multiplied by e^{-sz_0}, has the limit zero as $|z_0| \to \infty$. Thus $\omega(z_0, n)$ may be augmented by such a product. Consequently, result (14) is appropriate irrespective of the particular z selected. It is true, however, that the expansion ceases to be dominant for any z having the property that its real part $\leqq 0$.

In summary, the representation of $f(z)$ given in (2) follows immediately. In practice, k should be chosen as $[s/2]+1$, or perhaps, $[s/2]+2$, if it is desired to enlarge the sector for z over which the expansion is valid.

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[6] *Op. cit.*, p. 38.

CONFORMAL GEOMETRY OF HORN ANGLES OF HIGHER ORDER.*

By Mary Elizabeth Ladue.

Introduction. Under a conformal transformation a single curve has no invariant, but two intersecting curves have as an invariant the size of the angle between their tangents at their point of intersection. Since the order of contact is also a conformal invariant, two curves with first or higher order contact will be transformed into a pair of curves having the same order contact and the invariant mentioned above will have the trivial value zero. This configuration, an ordered pair of analytic curves with n-th order contact, is a *horn angle of the n-th order* with its *vertex* at their point of intersection. Kasner has shown that the lowest order non-trivial conformal invariant of a first order horn angle is a third order differential invariant. In fact he has shown that every horn angle has one and only one non-trivial conformal invariant which for an n-th order horn angle will be of differential order $2n + 1$. This $2n + 1$ order differential invariant of the n-th order horn angle he calls the *measure* of that horn angle, and he has found the expressions for the measures of first and second order horn angles (1). In this paper the measures of third, fourth and fifth order horn angles will be obtained.

The measure of an n-th order horn angle is determined by the values assumed by the first $2n + 1$ derivatives of the two curves forming the sides of the angle at their point of intersection. Consequently its formula can be expressed in terms of the first $2n + 2$ coefficients of the power series expansions of these curves (actually the first $n + 1$ of these cancel out), or in terms of the curvatures and arc-length derivatives of curvature evaluated at the vertex, or finally, in simplest form, in terms of $n + 1$ certain simple functions of these curvatures and their arc-length derivatives. For any particular curve the $n + 1$ values assumed by these functions at any particular point can be regarded as the coördinates of a point in an $n + 1$ dimensional space. If the set of all curves (analytic arcs) through a given point and having there n-th order contact with each other is called an *n-th order horn set*, then with such a horn set is associated the set of points corresponding to the curves of that set. This set of points is called the K_{n+1}*-space* associated with the given horn set. Since the distance between any two points of this space is given by the measure of the horn angle giving rise to these points, the study of the geometry

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of this space is equivalent to the study of the conformal invariants of this set of curves. The properties of the geometries of the K-spaces associated with first and second order horn angles have been studied in considerable detail by Kasner, de Cicco, and Comenetz (2), and in this paper the geometry of a third order horn set is investigated in order that the similarities and differences in the geometries corresponding to various ordered horn sets may be considered.

Roughly dual to the group of conformal transformations is the group of equilong transformations of the plane which convert straight lines into straight lines in such a way that the distance between the points of contact of any two curves (defined as the envelope of an infinitude of lines) having a common tangent remains invariant (3). When this distance is zero and the two points of contact are the same, the configuration becomes a horn angle which Kasner has shown to have a non-trivial equilong invariant of higher order (1a). As in the conformal case, it is possible to associate with each horn set an equilong geometry which will have as its metric the equilong measure of horn angles of that order of contact. Kasner has investigated the equilong geometry associated with first order horn sets and found it to be identical with that of the conformal geometry of first order horn sets (4). In this paper the equilong geometries corresponding to horn sets of second and third order are investigated and, as in the earlier case, are found to be identical with the corresponding conformal geometries. The comparison of the conformal and equilong geometries of several higher orders is to be published soon.

The problem of a conformal measure of a horn angle is interesting not only as the study of the invariants of an important geometry—the geometry associated with the group of conformal transformations of the plane—but historically in that it answers by modern methods a question the ancient Greek mathematicians were unable to cope with satisfactorily. For the conformal measure of a horn angle provides a means of distinguishing according to size between angles which the Greeks were unwillingly forced to call the same, a measure furthermore which is significant in that it is invariant under a group of transformations which includes the Euclidean group of rigid motions of Greek geometry.

1. Conformal measure of Horn angles of higher order. In this section the measures of horn angles of several higher orders of contact will be developed. Since these invariants can be expressed in terms of the metric invariants of curvature and the arc-length derivatives of curvature of the curves forming the sides of the angle, they are obviously invariant under the

group of rigid motions. Hence there is no loss in generality in considering the measure of the horn angle whose sides are the curves

$$(1) \quad y = a_2x^2 + a_3x^3 + \cdots$$

$$(1') \quad y = a'_2x^2 + a'_3x^3 + \cdots \quad (\text{with } a_2 = a'_2, \cdots, a_n = a'_n,\ a_{n+1} \neq a'_{n+1}),$$

which has the origin as its common point, the positive direction of the x-axis as its common direction, and has contact of order n at the origin. Furthermore, this study will be limited to the effect on this horn angle of those direct conformal transformations which carry the origin of the (x, y)-plane into the origin of the (X, Y)-plane and the positive direction on the x-axis into the positive direction on the X-axis. Such a transformation can, in general, be represented by a power series of the form

$$(2) \quad Z = c_1z + c_2z^2 + \cdots \qquad (\text{with } c_1 \text{ real and positive}),$$

or if $z = x + iy$, $Z = X + iY$, and $c_n = \alpha_n + i\beta_n$ by

$$(2') \quad \begin{aligned} X &= \alpha_1x + \alpha_2(x^2 - y^2) - 2\beta_2xy + \cdots \\ Y &= \alpha_1y + \beta_2(x^2 - y^2) + 2\alpha_2xy + \cdots \end{aligned} \qquad (\alpha_1 > 0).$$

Under the transformation $(2')$ the horn angle $(1, 1')$ is carried conformally into another horn angle of the same order of contact which may be given by the pair of curves

$$(3) \quad Y = A_2X^2 + A_3X^3 + \cdots$$

$$(3') \quad Y = A'^2X^2 + A'_3X^3 + \cdots \ (\text{with } A_2 = A'_2, \cdots, A_n = A'_n, A_{n+1} \neq A'_{n+1}).$$

The Measure of a Horn Angle of the Third Order. The measure of a horn angle of the third order will be derived by determining an expression in terms of the coefficients of the curves (1) and $(1')$ which is invariant under the transformation $(2')$. This will be done by first considering the expressions invariant under the modified form of the transformation which is obtained by setting $\alpha_1 = 1$,

$$(2'') \quad \begin{aligned} X &= x + \alpha_2(x^2 - y^2) - 2\beta_2xy + \cdots \\ Y &= y + \beta_2(x^2 - y^2) + 2\alpha_2xy + \cdots. \end{aligned}$$

This will cause a simplification in the calculations and a slight modification in the results which can easily be corrected later by following the transformation $(2'')$ by the simple magnification

$$(2''') \quad Z = \alpha_1z \qquad (\alpha_1 > 0).$$

The condition that the horn angle $(1, 1')$ be equivalent to the horn angle $(3, 3')$ under the transformation $(2'')$ is expressed by the two identities in x

$$(4)\quad (a_2x^2+a_3x^3+\cdots)$$
$$+\beta_2[x^2-(a_2x^2+a_3x^3+\cdots)^2]+2\alpha_2x(a_2x^2+a_3x^3+\cdots)+\cdots$$
$$\equiv A_2\{x+\alpha_2[x^2-(a_2x^2+a_3x^3+\cdots)^2]-2\beta_2x(a_2x^2+a_3x^3+\cdots)+\cdots\}^2+$$

$$(4')\quad (a'_2x^2+a'_3x^3+\cdots)$$
$$+\beta_2[x^2-(a'_2x^2+a'_3x^3+\cdots)^2]+2\alpha_2x(a'_2x^3+a'_3x^3+\cdots)+\cdots$$
$$\equiv A'_2\{x+\alpha_2[x^2-(a'_2x^2+a'_3x^3+\cdots)^2]-2\beta_2x(a'_2x^2+a'_3x^3+\cdots)+\cdots\}$$

obtained by substituting (2″) into (3) and (3′) and eliminating y by means of (1) and (1′). The following equations, representing the effect of the transformation (2″) on the coefficients of the curve (1), are obtained by equating to zero the coefficients of the various powers of x in the identity (4) and solving the resulting equations for $A_2,\cdots,A_7$:

$$(5_1)\quad A_2=a_2+\beta_2$$

$$(5_2)\quad A_3=a_3+\beta_3-2\alpha_2\beta_2$$

$$(5_3)\quad A_4=a_4-\alpha_2a_3+3\beta_2a_2^2+(\alpha_3-\alpha_2^2+4\beta_2^2)a_2+\beta_4-2\alpha_3\beta_2-3\alpha_2\beta_3+$$

$$(5_4)\quad A_5=a_5-2\alpha_2a_4+8\beta_2a_2a_3+(\alpha_2^2+4\beta_2^2)a_3+2\alpha_2a_2^3$$
$$+(3\beta_3-6\alpha_2\beta_2)a_2^2+(2\alpha_4-6\alpha_2\alpha_3+4\alpha_2^3-24\alpha_2\beta_2^2+12\beta_2\beta_3)a_2$$
$$+\beta_5-2\beta_2\alpha_4-4\alpha_2\beta_4-3\alpha_3\beta_3+12\alpha_2\beta_2\alpha_3+9\alpha_2^2\beta_3-14\alpha_2^3\beta_2$$

$$(5_5)^1\quad A_6=a_6-3\alpha_2a_5+\cdots+42\alpha_2^4\beta_2$$

$$(5_6)^1\quad A_7=a_7-4\alpha_2a_6+\cdots-132\alpha_2^5\beta_2.$$

The corresponding relations $(5'_1),\cdots,(5'_6)$ for the second side of the horn angle are obtained from these by replacing $a_2,\cdots,a_7$, $A_2,\cdots,A_7$ by $a'_2,\cdots,a'_7$, $A'_2,\cdots,A'_7$ respectively in the above equations.

In order to find the invariants, it is necessary to eliminate the coefficients α_i,β_i of the transformation. Since $\Delta a_2=\Delta a_3=0$,[2] the coefficients $\alpha_4,\alpha_5,\alpha_6$, $\beta_4,\beta_5,\beta_6,\beta_7$ can be eliminated by subtracting equations $(5_3),\cdots,(5_6)$ from the corresponding primed expressions. The resulting equations are

$$(6)\quad \Delta A_4=\Delta a_4$$

$$(7)\quad \Delta A_5=\Delta a_5-2\alpha_2\Delta a_4$$

$$(8)\quad \Delta A_6=\Delta a_6-3\alpha_2\Delta a_5+10\beta_2a_2\Delta a_4-(\alpha_3-4\alpha_2^2-4\beta_2^2)\Delta a_4$$

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[1] The complete expressions for A_6 and A_7 in terms of $a_2,\cdots,a_7$ and the coefficients of the transformation which involve, respectively, forty-five and eighty-three terms may be obtained from the author.

[2] Throughout this section of the paper the Δ notation will be used to indicate the difference between corresponding expressions connected with the two sides of a horn angle; e. g. $\Delta a_2=a'_2-a_2$, etc.

$$(9)\quad \begin{aligned}\Delta A_7 = {} & \Delta a_7 - 4\alpha_2\Delta a_6 + 12\beta_2 a_2\Delta a_5 - (2\alpha_3 - 8\alpha_2{}^2 - 4\beta_2{}^2)\Delta a_5 \\ & + 12\beta_2 a_3\Delta a_4 + 8\alpha_2 a_2{}^2\Delta a_4 + (12\beta_3 - 48\alpha_2\beta_2)a_2\Delta a_4 \\ & + (4\alpha_2\alpha_3 + 12\beta_2\beta_3 - 8\alpha_2{}^3 - 32\alpha_2\beta_2{}^2)\Delta a_4.\end{aligned}$$

Next β_2, β_3 are eliminated by substituting from equations (5_1) and (5_2) and α_2, α_3 by substituting from (7) and (8) into (9). Using (6) and rearranging, the expression

$$(10)\quad \begin{aligned}& \overline{\Delta A}_5{}^3 - 2\Delta A_6\Delta A_5\Delta A_4 + (\Delta A_7 + 4A_2{}^2\Delta A_5)\overline{\Delta A}_4{}^2 - 12A_2A_3\overline{\Delta A}_4{}^3 \\ & \quad = \overline{\Delta a}_5{}^3 - 2\Delta a_5\Delta a_5\Delta a_4 + (\Delta a_7 + 4a_2{}^2\Delta a_5)\overline{\Delta a}_4{}^2 - 12a_2a_3\overline{\Delta a}_4{}^3\end{aligned}$$

is obtained.

Equations (6) and (10) exhibit two quantities which are invariant under the special transformation (2″). Since the effect of the magnification (2‴) on the curve (1) is to divide the coefficient a_n by $\alpha_1{}^{n-1}$, equations (6) and (10) still hold under the general transformation (2′) provided the factors $1/\alpha_1{}^3$ and $1/\alpha_1{}^{12}$ are inserted in the right members of (6) and (10) respectively. ($\alpha_1 \neq 0$ for $\alpha_1{}^2$ is the Jacobian of the transformation (2′) at the origin.) Consequently, the quantities

$$(6')\quad \Delta a_4$$

$$(10')\quad \overline{\Delta a}_5{}^3 - 2\Delta a_6\Delta a_5\Delta a_4 + (\Delta a_7 + 4a_2{}^2\Delta a_5)\overline{\Delta a}_4{}^2 - 12a_2a_3\overline{\Delta a}_4{}^3$$

are relative invariants of the horn angle $(1, 1')$ under the conformal transformation (2′) of index 1 and 4 respectively, and

THEOREM I. *The quantity*

$$(11)\quad \frac{\overline{\Delta a}_4{}^4}{\overline{\Delta a}_5{}^3 - 2\Delta a_6\Delta a_5\Delta a_4 + (\Delta a_7 + 4a_2{}^2\Delta a_5)\overline{\Delta a}_4{}^2 - 12a_2a_3\overline{\Delta a}_4{}^3}$$

and the sign of Δa_4 are the invariants of a horn angle of the third order under the conformal transformation (2). *Conversely, if two such horn angles agree in the value of* (11) *and the sign of Δa_4, they can be made to coincide by a conformal transformation* (*at least to any finite order of contact*).

For the proof of the converse, see Kasner's Cambridge paper (1a). The quantity (11) will be called the *conformal measure of a third order horn angle* and will be denoted by $M_{12}{}^4$. It is a differential invariant of the seventh order.

The coefficients of the curve (1) are expressed in terms of the values, at the origin, of the curvature, γ_1, and its derivatives with respect to length of arc, $\gamma_1', \cdots, \gamma_1{}^v$ by the standard formulae

$$(12_1)\quad 2a_2 = \gamma_1$$

$$(12_2)\quad 6a_3 = \gamma_1'$$

$$(12_3)\quad 24a_4 = \gamma_1'' + 3\gamma_1{}^3$$

$$(12_4)\quad 120a_5 = \gamma_1''' + 19\gamma_1^2\gamma_1'$$

$$(12_5)\quad 720a_6 = \gamma_1^{iv} + 34\gamma_1^2\gamma_1'' + 48\gamma_1\gamma_1'^2 + 99\gamma_1^5$$

$$(12_6)\quad 5040a_7 = \gamma_1^{v} + 55\gamma_1^2\gamma_1''' + 199\gamma_1\gamma_1'\gamma_1'' + 48\gamma_1'^3 + 1135\gamma_1^4\gamma_1'$$

and the coefficients of the curve (1′) by similar formulae in $\gamma_2, \gamma_2', \cdots, \gamma_2^{v}$. Consequently, the invariant measure (11) can be written as follows:

$$(13)\quad M_{12}^4 = \frac{875}{4}\cdot\frac{\overline{\Delta\gamma''}^4}{42\overline{\Delta\gamma'''}^3 - 70\Delta\gamma''\Delta\gamma'''\Delta\gamma^{iv} + (25\Delta\gamma^{v} + 45\gamma^2\Delta\gamma''')\overline{\Delta\gamma''}^2 - 275\gamma\gamma'\overline{\Delta\gamma''}^3}.$$

This might be regarded as the intrinsic form for the conformal measure of a third order horn angle since it is invariant under any conformal transformation, but it is an unwieldly expression and it will be convenient to use a simpler form.

Let the equations

$$(14_1)\quad w = a_4$$

$$(14_2)\quad x = a_5$$

$$(14_3)\quad y = a_6$$

$$(14_4)\quad z = a_7 + 4a_2^2a_5 - 12a_2a_3a_4$$

define a correspondence between the number quadruples (w, x, y, z) and the coefficients $a_4, \cdots, a_7$ of the curve (1). Then the conformal measure of the third order horn angle can be written in the form

$$(15)\quad M_{12}^4 = \frac{\overline{\Delta w}^4}{\overline{\Delta x}^3 - 2\Delta x\Delta y\Delta w + \Delta z\overline{\Delta w}^2}$$

where w, x, y, z and w', x', y', z' can be expressed in terms of the metric invariants $\gamma_1, \gamma_1', \cdots, \gamma_1^{v}$ and $\gamma_2, \gamma_2', \cdots, \gamma_2^{v}$ by combining the equations (12) and (14).

The measure of a fourth order horn angle. The derivation of the conformal invariant of a horn angle of the fourth order is similar to the procedure outlined above for the third order case, presenting no new difficulties other than the increasing mechanical difficulty in handling the computation. The work can be somewhat simplified, however, by omitting the determination of the set of equations analogous to the set (5) and, instead, forming immediately a set of difference equations obtained by equating to zero the coefficient of x^n in the identity (4) and in the identity (4′), then subtracting the first equation thus obtained from the second, and finally solving the resulting difference equation for ΔA_n $(n = 5, \cdots, 9)$.

Note: These three operations, if done in this order, can be performed almost entirely by inspection of the two identities (4) and (4′) because the difference equations contain considerably fewer terms than the equations (5) due to the fact that in a fourth order horn angle $\Delta a_2 = \Delta a_3 = \Delta a_4 = 0$. However, the use of this method does entail the loss of the explicit statement of the effect of the transformation (2″) on the coefficients of the curve (1) which is needed in obtaining the group of induced transformations of a fourth order horn set and the expressions for β_8 and β_9 which would be needed in the elimination process in the derivation of the measures of horn angles of order greater than eight.

From these difference equations the coefficients α_i, β_i of the transformation can be eliminated as before. As a result of this elimination the quantities

(16) Δa_5

(17) $$\overline{\Delta a_6}^4 - 3\Delta a_7\overline{\Delta a_6}^2\Delta a_5 + (\overline{\Delta a_7}^2 + 2\Delta a_8\Delta a_6 + 4a_2{}^2\overline{\Delta a_6}^2)\overline{\Delta a_5}^2 - (\Delta a_9 + 4a_2{}^2\Delta a_7 + 12a_2a_3\Delta a_6)\overline{\Delta a_5}^3 + (9a_3{}^2 + 16a_2a_4)\overline{\Delta a_5}^4$$

are obtained as relative invariants of the fourth order horn angle formed by the curves $(1, 1')$ under the conformal transformation (2′). They are of index 1 and 5 respectively. Consequently,

THEOREM II. *The quantity*

(18) $$\frac{\overline{\Delta a_5}^5}{\overline{\Delta a_6}^4 - \cdots + (9a_3{}^2 + 16a_2a_4)\overline{\Delta a_5}^4}$$

and the sign of Δa_5 are the invariants of a fourth order horn angle under the conformal transformation (2). *Conversely, if two such horn angles agree in the value of* (18) *and in the sign of Δa_5, they can be made to coincide by a conformal transformations (at least to any finite order of contact).*

The quantity (18) will be called the *conformal measure of a fourth order horn angle* and denoted by $M_{12}{}^5$. It is a differential invariant of the ninth order.

Were it desirable, the quantities (16), (17), (18) could now be expressed in terms of the curvatures and arc-length derivatives of curvature of the curves of the horn angle as was done for the horn angle of third order contact in expression (13), but as this form is not needed, it will be omitted.

The measure, $M_{12}{}^5$, can be expressed in its simplified form analogous to (15) by means of the correspondence

(19$_1$) $v = a_5$

(19$_2$) $w = a_6$

(19$_3$) $x = a_7$

$$(19_4)\quad y = a_8 + 2a_2^2 a_6$$

$$(19_5)\quad z = a_9 + 4a_2^2 a_7 + 12a_2 a_3 a_6 - (9a_3^2 + 16a_2 a_4)a_5.$$

Then

$$(20)\quad M_{12}^5 = \frac{\overline{\Delta v}^5}{\overline{\Delta w}^4 - 3\Delta v\overline{\Delta w}^2\Delta x + (\overline{\Delta x}^2 + 2\Delta w\Delta y)\overline{\Delta v}^2 - \overline{\Delta v}^3\Delta z}$$

is the measure of the fourth order horn angle in terms of certain simple functions of the coefficients of the curves of the horn angle and consequently in terms of simple functions of the curvatures and arc-length derivatives of curvature of these curves at the origin. Thus M_{12}^5 is a metric invariant also.

The measure of a fifth order horn angle. Analogous to the results in the two preceding sections, the quantities

$$(21)\quad \Delta a_6$$

$$(22)\quad \begin{aligned}\overline{\Delta a_7}^5 &- 4\Delta\overline{a_7}^3\Delta a_8\Delta a_6 + (3\Delta\overline{a_7}^2\Delta a_9 + 3\Delta a_7\overline{\Delta a_8}^2 + 4a_2^2\overline{\Delta a_7}^3)\overline{\Delta a_6}^2\\ &- (2\Delta a_7\Delta a_{10} + 2\Delta a_8\Delta a_9 + 8a_2^2\Delta a_7\Delta a_8 + 12a_2a_3\overline{\Delta a_7}^2)\overline{\Delta a_6}^3\\ &+ [\Delta a_{11} + 4a_2^2\Delta a_9 + 12a_2a_3\Delta a_8 + (9a_3^2 + 16a_2a_4)\Delta a_7]\overline{\Delta a_6}^4\\ &- (20a_2a_5 + 24a_3a_4)\overline{\Delta a_6}^5\end{aligned}$$

are relative invariants of the horn angle of fifth order contact formed by the curves $(1, 1')$ under the conformal transformation $(2')$. They are of index 1 and 6 respectively. Consequently,

THEOREM III. *The quantity*

$$(23)\quad \frac{\overline{\Delta a_6}^6}{\overline{\Delta a_7}^5 - \cdots - \overline{\Delta a_6}^5(20a_2a_5 + 24a_3a_4)}$$

and the sign of Δa_6 are the invariants of a fifth order horn angle under the conformal transformation (2). Conversely, if two such horn angles agree in the value of (23) and the sign of Δa_6, they can be made to coincide (at least to any finite order of contact) by a conformal transformation.

The quantity (23) will be called the *conformal measure of the fifth order horn angle* and denoted by M_{12}^6. It is a differential invariant of the eleventh order.

Again the expression of these invariants in terms of the intrinsic metric invariants will be omitted. The measure, M_{12}^6, can be expressed in the simplified form

$$(24)\quad M_{12}^6 = \frac{\overline{\Delta u}^6}{\overline{\Delta v}^5 - 4\Delta u\overline{\Delta v}^3\Delta w + 3\overline{\Delta u}^2(\overline{\Delta v}^2\Delta x + \Delta v\overline{\Delta w}^2) - 2\overline{\Delta u}^3(\Delta v\Delta y + \Delta w\Delta x) + \overline{\Delta u}^4\Delta z}$$

by means of the correspondence

(25_1) $u = a_6$

(25_2) $v = a_7$

(25_3) $w = a_8$

(25_4) $x = a_9 + \frac{4}{3}a_2{}^2a_7$

(25_5) $y = a_{10} + \frac{8}{3}a_2{}^2a_8 + 6a_2a_3a_7$

(25_6) $z = a_{11} + 4a_2{}^2a_9 + 12a_2a_3a_8 + (9a_3{}^2 + 16a_2a_4)a_7 - (20a_2a_5 + 24a_3a_4)a_6.$

By combining equations (12_i) and (25_i), the functions u, v, w, x, y, z can be expressed in terms of the metric invariants, curvature and its arc-length derivatives, indicating that $M_{12}{}^6$ is a metric invariant also.

Since the existence of one and only one such conformal invariant has been established by Kasner for horn angles of any finite order of contact, and since the methods employed for each case are such simple extensions of the methods of the previous case, it seems probable that a general form for the invariant for the n-th order case exists. So far, however, it has not been possible to find a symbolism in which to express this conformal invariant of differential order $2n + 1$ which is sufficiently compact to permit a proof of its existence to be carried through by producing the measure.

2. Conformal geometry of third order Horn angles. In an earlier section of this paper it has been shown that two given horn angles of n-th order contact can be transformed into one another by a direct conformal transformation if and only if the two horn angles agree in measure (and in the sign of Δa_{n+1}) and that the measure of a given horn angle is completely determined by the values of the first $2n + 1$ derivatives of the sides at the vertex. Consequently, the study of the conformal geometry of horn angles may be reduced to the study of the conformal geometry of the differential elements of the $(2n + 1)$-st order passing through a given point in a given direction. In the remainder of this section the term *curve* will mean the differential element defined by a point on a given curve and the derivatives of this curve at that point, and the term n-th *order horn set* will mean the totality of all curves [$(2n + 1)$-st order elements] through a given point and each having at this point n-th order contact with every curve of the set. Again the n-th order horn set for which the common point and common direction are the origin and horizontal axis may be considered without loss of generality.

Associated with each third order horn set (γ, γ') (i. e., the horn set of curves through the origin having there slope zero, curvature γ, and arc-length

derivative of curvature γ') is a four dimensional space, $K_4(\gamma, \gamma')$, which consists of the totality of number quadruples (w, x, y, z) where w, x, y, z are the simple functions of the coefficients of the curves of the horn set (γ, γ') given by the equations $(14_i; \mathbf{1})$. The conformal transformation $(2; \mathbf{1})$ induces a transformation between the curves of the third order horn set (γ, γ') and those of another third order horn set (Γ, Γ') and, consequently, between the elements (w, x, y, z) of the space $K_4(\gamma, \gamma')$ and the elements (W, X, Y, Z) of the space $K_4(\Gamma, \Gamma')$. The geometry of the K_4-space associated with a given horn set (i. e., the invariants of this space under the transformation thus induced) will be the subject of this section of the paper.

Definitions of the elements of a k_4-space. A *point* of the K_4-space is a number quadruple (w, x, y, z) as defined by the equations $(14_i; \mathbf{1})$.

A *series* consists of any single infinitude of points of the space.

A *line* (linear series) consists of the single infinitude of points of the space which satisfy three linear equations of the form

$$x = p_1 w + r_1, \qquad y = p_2 w + r_2, \qquad z = p_3 w + r_3 \tag{1}$$

where p_i, r_i $(i = 1, 2, 3)$ are constants. When two lines are considered, double subscripts will be used, the second subscript indicating to which line the coefficient belongs.

A *plane* (flat congruence) consists of the double infinitude of points of the space which satisfy two linear equations of the form

$$y = a_1 w + b_1 x + c_1, \qquad z = a_2 w + b_2 x + c_2 \tag{2}$$

where a_i, b_i, c_i $(i = 1, 2)$ are constants. Second subscripts will be used to distinguish between the elements when more than one is considered.

A *flat* consists of the triple infinitude of points which satisfy the single linear equation

$$z = dw + fx + gy + k \tag{3}$$

where d, f, g, k are constants, subscripts indicating the flat to which they belong when more than one flat is considered.

Derivation of the fundamental group, G_7. The transformation induced by the conformal transformation $(2; \mathbf{1})$ and carrying the point (w, x, y, z) of the space $K_4(\gamma, \gamma')$ into the point (W, X, Y, Z) of the space $K_4(\Gamma, \Gamma')$ is given by the equations

$$\begin{aligned} W &= m_1^3 w + h_1 \\ X &= m_1^4 x + 2m_1^2 m_2 w + h_2 \\ Y &= m_1^5 y + 3m_1^3 m_2 x + m_1 m_3 w + h_3 \\ Z &= m_1^6 z + 4m_1^4 m_2 y + 2m_1^2 m_3 x + 4m_2(m_3 - 2m_2^2) w + h_4 \end{aligned} \tag{4}$$

where $m_1 \neq 0$, m_2, m_3, h_1, h_2, h_3, h_4 are constants. The verification of this set of equations lies in expressing the functions W, X, Y, Z, defined as functions of the coefficients of the transformed curve $(3;\mathbf{1})$, as functions of the coefficients of the curve $(1;\mathbf{1})$ and then expressing this last set of coefficients as functions of w, x, y, z. The effect of the transformation $(2'';\mathbf{1})$ on the coefficients of the curve $(1;\mathbf{1})$ is given by the equations $(5;\mathbf{1})$ from which β_2 and β_3 can be eliminated as before by the use of $(5_1;\mathbf{1})$ and $(5_2;\mathbf{1})$ leaving

$$
\begin{aligned}
A_4 &= a_4 + f(a_2, a_3, \alpha_i, \beta_i)\\
A_5 &= a_5 - 2\alpha_2 a_4 + g(a_2, a_3, \alpha_i, \beta_i)\\
A_6 &= a_6 - 3\alpha_2 a_5 + (-\alpha_3 + 4\alpha_2^2 + 4A_2^2 + 2A_2 a_2 - 6a_2^2)a_4\\
&\quad + h(a_2, a_3, \alpha_i, \beta_i)\\
A_7 &= a_7 - 4\alpha_2 a_6 + 2(-\alpha_3 + 4\alpha_2^2 + 2A_2^2 + 2A_2 a_2 - 4a_2^2)a_5\\
&\quad + [-4\alpha_2(-\alpha_3 + 2\alpha_2^2 + 2A_2^2 + 2A_2 a_2 - 6a_2^2)\\
&\quad + 12(A_2 A_3 - a_2 a_3)]a_4 + k(a_2, a_3, \alpha_i, \beta_i)
\end{aligned}
\tag{5}
$$

where f, g, h, k are functions of a_2, a_3, α_i, β_i $(i = 2, \cdots, 7)$ and do not involve $a_4, \cdots, a_7$. The effect of using the modified transformation $(2'';\mathbf{1})$ in place of the transformation $(2;\mathbf{1})$ is eliminated by dividing each of the coefficients a_n of the right members of equations (5) by α_1^{n-1} as before and the resulting equations,

$$
\begin{aligned}
A_4 &= (1/\alpha_1^3)a_4 + f'\\
A_5 &= (1/\alpha_1^4)a_5 - 2(\alpha_2/\alpha_1^3)a_4 + g'\\
A_6 &= (1/\alpha_1^5)a_6 - 3(\alpha_2/\alpha_1^4)a_5 + (1/\alpha_1^5)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)a_4 + h'\\
A_7 &= (1/\alpha_1^6)a_7 - 4(\alpha_2/\alpha_1^5)a_6 + 2(1/\alpha_1^6)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 4a_2^2)a_5\\
&\quad + [-4(\alpha_2/\alpha_1^5)(-\alpha_1^2\alpha_3 + 2\alpha_1^2\alpha_2^2 + 2\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)\\
&\quad + 12(1/\alpha_1^6)(\alpha_1^3 A_2 A_3 - a_2 a_3)]a_4 + k',
\end{aligned}
\tag{6}
$$

express the effect of a more general transformation like $(2;\mathbf{1})$ on the coefficients of the curve $(1;\mathbf{1})$. From these equations (6), the equations $(14_i;\mathbf{1})$ and the corresponding equations expressing W, X, Y, Z as functions of $A_4, \cdots, A_7$, are obtained the equations

$$
\begin{aligned}
W &= (1/\alpha_1^3)w + f'\\
X &= (1/\alpha_1^4)x - 2(\alpha_2/\alpha_1^3)w + g'\\
Y &= (1/\alpha_1^5)y - 3(\alpha_2/\alpha_1^4)x + (1/\alpha_1^5)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)w + h'\\
Z &= (1/\alpha_1^6)z - 4(\alpha_2/\alpha_1^5)y + 2(1/\alpha_1^6)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)x\\
&\quad - 4(\alpha_2/\alpha_1^5)(-\alpha_1^2\alpha_3 + 2\alpha_1^2\alpha_2^2 + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)w\\
&\quad + k' + 4A_2^2 g' - 12A_2 A_3 f'
\end{aligned}
\tag{7}
$$

which become the equations of the induced transformation (4) when the substitutions

$$(8)\quad \begin{aligned} &m_1 = 1/\alpha_1,\ m_2 = -\alpha_2/\alpha_1,\ m_3 = (1/\alpha_1^4)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2 + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_3 g_1 \\ &h_1 = f',\ h_2 = g',\ h_3 = h',\ h_4 = k' + 4A_2^2 g' - 12A_2 A_3 f' \end{aligned}$$

are made. Consequently,

THEOREM IV. *The group of conformal transformations of the plane induces a seven parameter group, G_7, of transformations between the points of the two K_4-spaces associated with the given and transformed horn sets.*

Furthermore, Theorem I may be restated as follows

THEOREM V. *An ordered pair of points of a space $K_4(\gamma, \gamma')$ has the unique absolute conformal invariant, the measure or distance between the points, M_{12}^4 (15; 1).*

Thus the differential of arc in this space is

$$(9)\quad ds = \frac{dw^4}{dw^2 dz - 2dwdxdy + dx^3}.$$

The general line. A *null series* shall mean any series of the K_4-space for which the measure of any pair of distinct points of the series is either zero or infinity. Thus from (9) it is seen that the null series are solutions of the differential equation (prime indicating differentiation with respect to w)

$$(10)\quad z' - 2x'y' + x'^3 = 0$$

and the special lines for which

$$(11)\quad w = \text{constant}.$$

The line (1) is said to be a *general line* if

$$(12)\quad p_3 - 2p_1p_2 + p_1^3 \neq 0.$$

Geometrically, a general line is a line which is not tangent to a null series (i. e., the line and series do not have two consecutive points in common).

Any general line (1) of a space $K_4(\gamma, \gamma')$ is converted into a general line

$$(1')\quad X = P_1W + R_1, \qquad Y = P_2W + R_2, \qquad Z = P_3W + R_3$$

of a space $K_4(\Gamma, \Gamma')$ under transformations of the fundamental group. This statement is proved by obtaining the effect of the transformation (4) on the coefficients of the line (1) which is given by the equations

$$
\begin{aligned}
&P_1 = m_1 p_1 + 2m_2/m_1 \\
&P_2 = m_1^2 p_2 + 3m_2 p_1 + m_3/m_1^2 \\
&P_3 = m_1^3 p_3 + 4m_1 m_2 p_2 + 2m_3 p_1/m_1 + (4m_2/m_1^3)(m_3 - 2m_2^2) \\
&R_1 = m_1^4 r_1 - m_1 h_1 p_1 + h_2 - 2m_2 h_1/m_1 \\
&R_2 = m_1^5 r_2 + 3m_1^3 m_2 r_1 - m_1^2 h_1 p_2 - 3m_2 h_1 p_1 + h_3 - m_3 h_1/m_1^2 \\
&R_3 = m_1^6 r_3 + 4m_1^4 m_2 r_2 + 2m_1^2 m_3 r_1 - m_1^3 h_1 p_3 - 4m_1 m_2 h_1 p_2 \\
&\qquad - 2m_3 h_1 p_1/m_1 + h_4 - (4m_2/m_1^3)(m_3 - 2m_2^2) h_1 .
\end{aligned} \tag{13}
$$

Then it can be seen that

$$
P_3 - 2P_1P_2 + P_1^3 = m_1^3 (p_3 - 2p_1p_2 + p_1^3), \tag{14}
$$

and consequently the line (1′) is a general line if the line (1) is.

Given two general lines, l_1 and l_2, it is possible to obtain from the equations (13) the following expressions from which the coefficients m_2, m_3, $h_1, \cdots, h_4$ of the transformation (4) have been eliminated [3]

$$
\begin{aligned}
&\Delta P_1 = m_1 \Delta p_1 \\
&\Delta P_2 \Delta R_1 - \Delta P_1 \Delta R_2 = m_1^6 (\Delta p_2 \Delta r_1 - \Delta p_1 \Delta r_2) \\
&3\Delta P_1 (\Delta P_3 \Delta R_1 - \Delta P_1 \Delta R_3) - 4\Delta P_2 (\Delta P_2 \Delta R_1 - \Delta P_1 \Delta R_2) \\
&\qquad = m_1^8 [3\Delta p_1 (\Delta p_3 \Delta r_1 - \Delta p_1 \Delta r_3) - 4\Delta p_2 (\Delta p_2 \Delta r_1 - \Delta p_1 \Delta r_2)] .
\end{aligned} \tag{15}
$$

The left sides of these equations (15) and of the two expressions (14) stated once for each line are then relative invariants, consequently,

THEOREM VI. *Two* [4] *general lines possess the four independent absolute invariants*

$$
\frac{\overline{\Delta p_1}^3}{p_{3j} - 2p_{2j}p_{1j} + p_{1j}^3} \qquad (j = 1, 2) \tag{16_j}
$$

$$
\frac{\Delta p_2 \Delta r_1 - \Delta p_1 \Delta r_2}{(p_{31} - 2p_{11}p_{21} + p_{11}^3)^2} \tag{16_3}
$$

$$
\frac{3\Delta p_1 (\Delta p_3 \Delta r_1 - \Delta p_1 \Delta r_3) - 4\Delta p_2 (\Delta p_2 \Delta r_1 - \Delta p_1 \Delta r_2)}{(p_{31} - 2p_{11}p_{21}p_{11}^3)^{8/3}} . \tag{16_4}
$$

Note: Two intersecting general lines possess only the two independent invariants (16_j).

The *direct dihorn angle*, α_{ij}, between the lines l_i and l_j is defined to be the quantity

---

[3] Throughout this section of the paper the second set of subscripts will be used to indicate to which of two elements the coefficient belongs and the Δ notation to indicate the difference between corresponding coefficients for the two elements.

[4] A single line possesses no absolute invariants but has the expression (12) as a relative invariant.

$$(17) \qquad \frac{(p_{1j}-p_{1i})^3}{p_{3j}-2p_{1j}p_{2j}+p_{1j}^2}\ ;$$

and the *reverse dihorn angle*, $\varkappa_{ji}$, for this ordered pair of lines, is defined by formula (17) with the i and the j interchanged. From the definition of these two angles and definition (15; I) the trihornometry (2b) of the third order can be developed.

Similar to the proof of Theorem VI is the proof of

THEOREM VII. *A point* (w_0, x_0, y_0, z_0) *and the general line* (1), *not containing the general point, have the three independent absolute invariants*

$$(18_1) \quad I_{Pl_1} = \frac{x_0 - p_1 w_0 - r_1}{(p_3 - 2p_1p_2 + p_1^3)^{4/3}}$$

$$(18_2) \quad I_{Pl_2} = \frac{2(y_0 - p_2 w_0 - r_2) - 3p_1(x_0 - p_1 w_0 - r_1)}{(p_3 - 2p_1p_2 + p_1^3)^{5/3}}$$

$$(18_3) \quad I_{Pl_3} = \frac{(z_0 - p_3 w_0 - r_3) - 2p_1(y_0 - p_2 w_0 - r_2) + (3p_1^2 - 2p_2)(x_0 - p_1 w_0 - r_1}{(p_3 - 2p_1p_2 + p_1^3)^2}$$

The extremum distance between the point and the line is a function of these invariants

$$(18_4) \qquad \text{Extremum } D = \frac{W^4}{W^3 - I_{Pl_3}W^2 - I_{Pl_1}I_{Pl_2}W - I_{Pl_1}^3}$$

where W is that function of $I_{Pl_1}, I_{Pl_2}, I_{Pl_3}$ obtained by solving the cubic equation

$$(19) \qquad W^3 - 2I_{Pl_3}W^2 - 3I_{Pl_1}I_{Pl_2}W - 4I_{Pl_1}^3 = 0.$$

The general plane. The plane (2) is a *general plane* if

$$(20) \quad [27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2 - 4[3(2a_1 - b_2) + 4b_1^2]^3 \neq 0.$$

Geometrically a general plane is a plane which is not the osculating plane of any null series (i. e., the plane and series do not have three consecutive points in common).

From equations (4) it is possible to show that under the fundamental group, G_7, any general plane (2) of a space $K_4(\gamma, \gamma')$ is converted into a general plane of a space $K_4(\Gamma, \Gamma')$

$$(2') \qquad Y = A_1W + B_1X + C_1, \qquad Z = A_2W + B_2X + C_2.$$

The transformation between the planes of the two K_4-spaces is given by the equations

$$
\begin{aligned}
&A_1 = m_1^2 a_1 - 2m_2 b_1 + (m_3 - 6m_2^2)/m_1^2 \\
&B_1 = m_1 b_1 + 3m_2/m_1 \\
&C_1 = m_1^5 c_1 + (2m_2 h_1 - m_1 h_2) b_1 - m_1^2 h_1 a_1 + h_3 - 3m_2 h_2/m_1 - (m_3 - 6m_2^2) h_1/m_1^2 \\
&A_2 = m_1^3 a_2 - 2m_1 m_2 b_2 + 4m_1 m_2 a_1 - 8m_2^2 b_1/m_1 - 8m_2^3/m_1^3 \\
&B_2 = m_1^2 b_2 + 4m_2 b_1 + 2m_3/m_1^2 \\
&C_2 = m_1^6 c_2 + m_1 (2m_2 h_1 - m_1 h_2) b_2 - m_1^3 h_1 a_2 + 4m_1^4 m_2 c_1 \\
&\quad + (4m_2/m_1)(2m_2 h_1 - m_1 h_2) b_1 - 4m_1 m_2 h_1 a_1 + h_4 - 2m_3 h_2/m_1^2 + 8m_2^3 h_1/m_1^3
\end{aligned} \tag{21}
$$

From equations (21) it is possible to obtain the following invariants:

THEOREM VIII. *A general plane* (2) *possesses the absolute invariant*

$$
\frac{[27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2}{4[3(2a_1 - b_2) + 4b_1^2]^3} \neq 1. \tag{22}
$$

THEOREM IX. *Two general planes possess, in addition to the invariant* (22) *stated for each plane, the four independent absolute invariants*

$$
\frac{[3(2a_{12} - b_{22}) + 4b_{12}^2]^3}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3} \tag{23_1}
$$

$$
\frac{\overline{\Delta b_1}^6}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3} \tag{23_2}
$$

$$
\frac{(2\Delta a_1 + \Delta b_2)^3}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3} \tag{23_3}
$$

$$
\frac{(\Delta a_2 \Delta b_1 - \Delta a_1 \Delta b_2)^{3/2}}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3}. \tag{23_4}
$$

Note: In general two planes have only a single point in common, but if they have a line in common, the invariant (23_4) vanishes.

THEOREM X. *A point* (w_0, x_0, y_0, z_0) *and a general plane* (2) *not containing the given point possess, in addition to the invariant* (22) *of the plane, the two independent absolute invariants*

$$
\frac{(y_0 - a_1 w_0 - b_1 x_0 - c_1)^{6/3}}{[27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2 - 4[3(2a_1 - b_2) + 4b_1^2]^3} \tag{24_1}
$$

$$
\frac{3(z_0 - a_2 w_0 - b_2 x_0 - c_2) - 4b_1(y_0 - a_1 w_0 - b_1 x_0 - c_1)}{[27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2 - 4[3(2a_1 - b_2) + 4b_1^2]^3}. \tag{24_2}
$$

THEOREM XI. *A general line* (1) *and a general plane* (2) *possess, in addition to the invariant* (22) *of the plane, the four independent absolute invariants*

$$
\frac{27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3}{p_3 - 2p_1 p_2 + p_1^3} \tag{25_1}
$$

$$(25_2)\qquad \frac{(2b_1-3p_1)^3}{p_3-2p_1p_2+p_1^3}$$

$$(25_3)\qquad \frac{(a_1+p_1b_1-p_2)^3}{(p_3-2p_1p_2+p_1^3)^2}$$

$$(25_4)\qquad \frac{[(a_2+p_1b_2-p_3)(c_1+r_1b_1-r_2)-(a_1+p_1b_1-p_2)(c_2+r_1b_2-r_3)]^2}{(p_3-2p_1p_2+p_1^3)^3}$$

Note: If the line (1) lies in the plane (2), the invariants (25_3) and (25_4) vanish.

The general flat. The flat (3) is said to be a *general flat* if

$$(26)\qquad 8d+4fg+g^3\neq 0.$$

Such a flat is not the osculating flat of a null series (i. e., the flat and the series do not have four consecutive points in common) at a point for which $x'=g/2$.

From equations (4) it is possible to show that under the fundamental group, G_7, any general flat (3) of a space $K_4(\gamma,\gamma')$ is converted into a general flat of a space $K_4(\Gamma,\Gamma')$

$$(3')\qquad Z=DW+FX+GY+K.$$

The transformation between the flats of the two K-spaces is given by the equations

$$(27)\qquad \begin{aligned} D&=m_1^3d-2m_1m_2f-(m_3-6m_2^2)g/m_1-(4m_2/m_1^3)(m_3-4m_2^2)\\ F&=m_1^2f-3m_2g+(2/m_1^2)(m_3-6m_2^2)\\ G&=m_1g+4m_2/m_1\\ K&=m_1^6k-m_1^3h_1d+m_1(2m_2h_1-m_1h_2)f\\ &\quad+[(m_3-6m_2^2)h_1/m_1+3m_2h_2-m_1h_3]g+h_4-4m_2h_3/m_1\\ &\quad-(2/m_1^2)(m_3-6m_2^2)h_2+(4m_2/m_1^3)(m_3-4m_2^2)h_1. \end{aligned}$$

From equations (27) it is possible to obtain the invariants listed in the following theorems.

THEOREM XII. *Two*[5] *general flats possess the two independent absolute invariants*

$$(28_j)\qquad \frac{\overline{\Delta g^3}}{8d_j+4f_jg_j+g_j^3}\qquad (j=1,2).$$

THEOREM XIII. *A point* (w_0,x_0,y_0,z_0) *and a general flat* (3) *not containing the given point possess the absolute invariant*

[5] A single flat possesses no absolute invariant but has the expression (26) as a relative invariant.

$$\frac{256(z_0 - dw_0 - fx_0 - gy_0 - k)}{(8d + 4fg + g^3)^2} \tag{29}$$

which is the extremum distance between the point and the flat.

THEOREM XIV. *A general line* (1) *and a general flat* (3) *possess the three independent absolute invariants*

$$\frac{(2p_1 - g)^3}{8d + 4fg + g^3} \tag{30_1}$$

$$\frac{(2p_1 - g)^3}{p_3 - 2p_1p_2 + p_1^3} \tag{30_2}$$

$$\frac{p_3 - d - fp_1 - gp_2}{8d + 4fg + g^3}. \tag{30_3}$$

Note: In general a line and a flat have a single point in common, in which case

$$p_3 - d - fp_1 - gp_2 \neq 0 \tag{31}$$

and no extremum distance can be computed. If in particular the line and the flat do not intersect (i. e., the line is " parallel " to the flat), the expression (31) vanishes but

$$r_3 - k - fr_1 - gr_2 \neq 0 \tag{32}$$

and the extremum distance is given by the expression

$$\text{Extremum } D = \frac{256(r_3 - k - fr_1 - gr_2)}{(8d + 4fg + g^3)^2} \tag{33}$$

which in this case *only* is an absolute invariant. If on the other hand the line lies entirely within the flat, both (31) and (32) vanish and the expression (33) gives the extremum distance, which is then zero.

THEOREM XV. *A general plane* (2) *and a general flat* (3) *possess, in addition to the invariant* (22) *of the plane, the three independent absolute invariants*

$$\frac{27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3}{8d + 4fg + g^3} \tag{34_1}$$

$$\frac{(4b_1 - 3g)^3}{8d + 4fg + g^3} \tag{34_2}$$

$$\frac{[3(2a_1 - f) + b_1(4b_1 - 3g)]^3}{(8d + 4fg + g^3)^2}. \tag{34_3}$$

Note: As in Theorem XIV, no extremum distance can be computed if the plane and flat intersect either in a line or in a point; but if they do not intersect

$$a_2 - d - a_1 g = b_2 - f - b_1 g = 0 \tag{35}$$

and the extremum distance is given by the expression

$$\text{Extremum } D = \frac{256(c_2 - c_1 g - k)}{(8d + 4fg + g^3)^2} \tag{36}$$

which in this case *only* is an absolute invariant, or if the flat contains the plane both (35) and (36) vanish, the extremum distance being zero.

3. Equilong geometry of second and third order Horn angles.

The equilong measure of second and third order horn angles. In this section of this paper the equilong measure and the equilong geometry of second and third order horn angles will be developed in order that the rough duality existing between the conformal and the equilong theories may be studied further.

An equilong transformation of the plane is any contact transformation of the plane which converts straight lines into straight lines in such a way that the distance, δ, between the points of contact of any two curves having a common tangent remains invariant. Since the element is the straight line, it will be convenient to use Hessian line coördinates (u, v) to indicate that directed line having a normal through the origin which is v units long and makes an angle of u radians with the prime direction. In this coördinate system a curve is given as the envelope of a single infinitude of lines and therefore by an equation of the form

$$v = a_0 + a_1 u + a_2 u^2 + \cdots.$$

Two such curves will have contact of exactly n-th order at the line $v = a_0$ if the first $n + 1$ coefficients in their power series expansions are the same and the $(n + 2)$-nd are distinct. If these two curves have contact of first or higher order, the invariant distance, δ, has the trivial value zero and the pair of curves form a horn angle of the given order of contact. This horn angle will have a unique equilong invariant. The form of this invariant for second and third order horn angles is obtained below. As in the conformal case there is no loss in generality in limiting the study to the invariants of the horn angle

$$(1) \quad \begin{aligned} &v = a_2 u^2 + a_3 u^3 + \cdots \\ &v = a'_2 u^2 + a'_3 u^3 + \cdots \quad (\text{with } a_2 = a'_2, \cdots, a_n = a'_n,\ a_{n+1} \neq a'_{n+1}). \end{aligned}$$

Since the group of equilong transformations of the plane can be shown to consist of the set of all line transformations of the form

$$\begin{aligned} U &= \phi(u) \\ V &= \phi_u(u) v + \psi(u) \end{aligned}$$

where ϕ and ψ are arbitrary analytic functions (3), the equations

$$(2)\qquad \begin{aligned} U &= \alpha_1 u + \alpha_2 u^2 + \cdots \\ V &= (\alpha_1 + 2\alpha_2 u + \cdots)v + \beta_1 u + \beta_2 u^2 + \cdots \qquad (\alpha_1 \neq 0) \end{aligned}$$

represent an adequately general equilong transformation under which the horn angle (1) is equivalent to another horn angle of the same order of contact given by the equations

$$(3)\qquad \begin{aligned} V &= A_2 U^2 + A_3 U^3 + \cdots \\ V &= A'_2 U^2 + A'_3 U^3 + \cdots \text{ (with } A_2 = A'_2, \cdots, A_n = A'_n,\ A_{n+1} \neq A'_{n+1}). \end{aligned}$$

Again the two identities

$$(4)\qquad \begin{aligned} &(\alpha_1 + 2\alpha_2 u + \cdots)(a_2 u^2 + a_3 u^3 + \cdots) + \beta_1 u + \beta_2 u^2 + \cdots \\ &\quad \equiv A_2(\alpha_1 u + \alpha_2 u^2 + \cdots)^2 + A_3(\alpha_1 u + \alpha_2 u^2 + \cdots)^3 + \cdots \end{aligned}$$

$$(4')\qquad \begin{aligned} &(\alpha_1 + 2\alpha_2 u + \cdots)(a'_2 u^2 + a'_3 u^3 + \cdots) + \beta_1 u + \beta_2 u^2 + \cdots \\ &\quad \equiv A'_2(\alpha_1 u + \alpha_2 u^2 + \cdots)^2 + A'_3(\alpha_1 u + \alpha_2 u^2 + \cdots)^3 + \cdots \end{aligned}$$

are the conditions for the equivalence of the horn angles (1) and (3) under the transformation (2). As before, when for a horn angle of a given order the coefficients of like powers of u are equated to zero in each identity, difference equations formed from the corresponding pairs of equations, and the constants α_i and β_i of the transformation (2) eliminated, a pair of relative invariants are obtained for this horn angle. For the second order horn angle the relative invariants are

$$(5)\qquad \begin{aligned} &\Delta a_3 && \text{of index 2} \\ &\overline{\Delta a_4}^2 - \Delta a_3 \Delta a_5 && \text{of index 6} \end{aligned}$$

and for the third order horn angle

$$(6)\qquad \begin{aligned} &\Delta a_4 && \text{of index 3} \\ &\overline{\Delta a_5}^3 - 2\Delta a_4 \Delta a_5 \Delta a_0 + \overline{\Delta a_4}^2 \Delta a_7 && \text{of index 12} \end{aligned}$$

Consequently,

THEOREM XVI. *A horn angle of second order has the unique absolute equilong invariant*

$$(5')\qquad \frac{\overline{\Delta a_3}^3}{\overline{\Delta a_4}^2 - 2\Delta a_3 \Delta_5},$$

and a horn angle of third order has the unique absolute equilong invariant

$$(6')\qquad \frac{\overline{\Delta a_4}^4}{\overline{\Delta a_5}^3 - 2\Delta a_4 \Delta a_5 \Delta a_6 + \overline{\Delta a_4}^2 \Delta a_7}.$$

These quantities (5′) and (6′) are called the *equilong measures of second and third order horn angles* respectively and are denoted by $\mu_{12}{}^3$ and $\mu_{12}{}^4$.

As in the conformal theory, the invariants can be expressed in terms of intrinsic metric invariants, in this case in terms of the radii of curvature and their derivatives with respect to the inclinations of the curves.[6] However, it is more desirable for comparing the equilong and conformal measures to leave the equilong measures expressed in terms of the power series coefficients, regarding these as functions of the metric invariants of radius of curvature and its inclination-derivatives. Then the comparison of the equilong measures $\mu_{12}{}^3$ (5′) and $\mu_{12}{}^4$ (6′) with the corresponding conformal measures $M_{12}{}^3$ [6] and $M_{12}{}^4$ (15; **1**) reveals that the two measures for second and third order horn angles are identical in algebraic structure.

The "duality" of the conformal and equilong geometries of second and third order horn angles. In section **2** of this paper the transformation induced in the space of elements (w, x, y, z) associated with a third order horn set by the conformal transformation (2; **1**) has been discussed in detail. The conformal geometries associated with first and second order horn sets has also been discussed thoroughly by Kasner and de Cicco (2). Furthermore Kasner has shown that the equilong geometry associated with a first order horn set is identical with the conformal geometry associated with that horn set (4b). In this section of this paper the extension of this "duality" to the geometries of second and third order horn angles is established.

The conformal K_3-space associated with a given second order horn set may be defined as the space of elements (x, y, z) given by the equations (2c, d)

$$(7)\qquad \begin{aligned} x &= a_3 \\ y &= a_4 \\ z &= a_5 - 4a_2{}^2a_3 \end{aligned}$$

where the a_i are the coefficients in the power series expansion (1; **1**) of the curves of the second order horn set. For this space the fundamental transformation induced by the conformal transformation (2; **1**) of the Cartesian plane is given by the equations [7]

---

[6] $\mu_{12}{}^3 = \dfrac{40}{3}\,\dfrac{\overline{\Delta r'}{}^3}{5\overline{\Delta r''}{}^2 - 4\Delta r'\Delta r''' + 4\overline{\Delta r'}{}^2}$;

$\mu_{12}{}^4 = \dfrac{875}{4}\,\dfrac{\overline{\Delta r''}{}^4}{42\overline{\Delta r'''}{}^3 - 70\Delta r''\Delta r'''\Delta r^{iv} + (25\Delta r^v + 45\Delta r''')\overline{\Delta r''}{}^2}$; $M_{12}{}^3 = \dfrac{\overline{\Delta x}{}^3}{\overline{\Delta y}{}^2 - \Delta x\Delta}$

[7] Equations (7) and (8) differ a little from the definition and transformation of the K-space given by Kasner and de Cicco because the author found this definition more convenient. The effect on the geometry is unimportant.

$$
\begin{aligned}
X &= m_1^2 x + h_1 \\
Y &= m_1^3 y + m_1 m_2 x + h_2 \\
Z &= m_1^4 z + 2m_1^2 m_2 y + m_2^2 x + h_3
\end{aligned} \tag{8}
$$

and the fundamental invariant distance under this transformation (i. e., $M_{12}{}^3$, the conformal measure of a second order horn angle) by

$$
\frac{\overline{\Delta x}^3}{\overline{\Delta y}^2 - \Delta x \Delta z}. \tag{9}
$$

If the equilong K'_4-space associated with the second order horn set is defined as the space of elements (x, y, z) given by the equations

$$
x = a_3, \quad y = a_4, \quad z = a_5, \tag{10}
$$

where the a_i are the coefficients in the power series expansion (1) of the curves of the horn set in Hessian line coördinates, then the transformation induced in this equilong K'_4-space by the equilong transformation (2) of the Hessian plane is also given by the equations (8) and the fundamental distance invariant, the equilong measure of a second order horn angle, by equation (9). Thus the "duality" extends to second order horn angles.

In section **2** of this paper the conformal K_4-space associated with a third order horn set was defined as the space of elements (w, x, y, z) given by the equations $(14_i; \mathbf{1})$ with an induced transformation given by the equations $(4; \mathbf{2})$ and a metric given by the expression $(15; \mathbf{1})$. In this paragraph the corresponding equilong K'_4-space will be defined and shown to be identical with this conformal K_4-space. The *equilong K'_4-space* associated with a given third order horn set is defined to be the space of elements (w, x, y, z) given by the equations

$$
w = a_4, \quad x = a_5, \quad y = a_6, \quad z = a_7, \tag{11}
$$

where the a_i are the coefficients in the power series expansion (1) of the curves of the horn set in Hessian line coördinates. The development of the equations of the transformation induced in this space by an equilong transformation (2) of the Hessian plane follows the same procedure as that outlined on pages 18 and 19 for the conformal case. Given a third order horn angle expressed in Hessian line coördinates in the form (1), it is possible to obtain from the identity (4) the set of equations

$$
\begin{aligned}
A_4 &= (1/\alpha_1{}^3) a_4 + h_1(a_2, a_3, \alpha_i) \\
A_5 &= (1/\alpha_1{}^4) a_5 - 2(\alpha_2/\alpha_1{}^5) a_4 + h_2(a_2, a_3, \alpha_i) \\
A_6 &= (1/\alpha_1{}^5) a_6 - 3(\alpha_2/\alpha_1{}^6) a_5 - (1/\alpha_1)(\alpha_3/\alpha_1{}^5 - 4\alpha_2{}^2/\alpha_1{}^6) a_4 + h_3(a_2, a_3, \alpha_i) \\
A_7 &= (1/\alpha_1{}^6) a_7 - 4(\alpha_2/\alpha_1{}^7) a_6 - 2(1/\alpha_1{}^2)(\alpha_3/\alpha_1{}^5 - 4\alpha_2{}^2/\alpha_1{}^6) a_5 \\
&\quad + 4(\alpha_2/\alpha_1{}^3)(\alpha_3/\alpha_1{}^5 - 2\alpha_2{}^2/\alpha_1{}^6) a_4 + h_4(a_2, a_3, \alpha_i)
\end{aligned} \tag{12}
$$

(where $h_1, \cdots, h_4$ are functions of a_2, a_3, α_i and do not involve $a_4, \cdots, a_7$). These equations, which express the effect of the equilong transformation (2) on the coefficients of the curve (1), become the equations of the desired transformation by means of (11) when the substitutions

$$(13) \qquad m_1 = 1/\alpha_1, \quad m_2 = -\alpha_2/\alpha_1^3, \quad m_3 = -[\alpha_3/\alpha_1^5 - 4(\alpha_2^2/\alpha_1^6)]$$

are made. It is only necessary to write out this transformation in order to see that it is given by the equations (4; **2**) and consequently has the expression (**15**; **1**) as its fundamental invariant. Consequently,

THEOREM XVII. *The equilong geometry of a second or third order horn set is identical with the conformal geometry of that horn set.*

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ON THE SINGULARITIES AND THE QUALITATIVE BEHAVIOR OF THE SOLUTIONS OF THE PROBLEM OF CENTRAL FORCES.*

By E. K. Haviland.

Introduction. The purpose of the present note is to investigate the motion of two bodies under the influence of their mutual attractions, when these latter satisfy certain specific conditions. By choosing the origin of the coördinate system so that it coincides with one of the particles, the motion may be reduced essentially to that of a single particle under a static central force. The force function, U, will in addition be subjected to the requirement that it be homogeneous of degree λ ($\neq 0$) in the Cartesian coördinates x and y. In the particular case where $\lambda = 0$ the latter requirement is replaced by the condition that the first partial derivatives U_x, U_y be homogeneous in x, y of degree $\lambda - 1 = -1$. This leads in the former case to a force function U proportional to r^λ and in the latter to one proportional to $\log r$, where the constant of proportionality is chosen so as to correspond to an attractive force.[1]

While a number of the results concerning the nature of the paths in various particular cases have long been known, the present paper is concerned primarily with an aspect of the subject which appears not to have been treated in detail hitherto, viz., the analytic character of the collisions, a subject which appears elementary at first glance, but which turns out to be complicated enough to involve a rather deep analytical apparatus; in particular, the proof of the non-existence of an analytic continuation in certain cases will be made to depend on Bohr's theory of almost-periodic functions of a complex variable.

At the same time, it has seemed desirable to complete and systematize the description of all the various cases arising from force functions of the above type, and in this connection the most striking result is that in the range $-2 < \lambda < 0$ the situation is the same as for the Newtonian exponent $\lambda = -1$; in the sense that the non-vanishing of the constant angular momentum, c, remains necessary and sufficient for the non-existence of collisions

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* Received March 25, 1942.

[1] Cf. W. Whewell, *On the Free Motion of Points*, Cambridge (1836), especially pp. 39-79. E. T. Whittaker, *Analytical Dynamics*, Cambridge (1917), pp. 77-86. P. Stäckel, *Encyclopädie der mathematischen Wissenschaften*, vol. 4, pp. 494-496.

and singularities, and that in addition the three subcases belonging to a fixed $c \neq 0$ are qualitatively the same as in the classical case.

1. Classification of cases. Turning now to the detailed treatment of the problem, we observe that, if polar coördinates be introduced, the equations of motion possess the integrals

$$\tfrac{1}{2}(r'^2 + r^2\phi'^2) - U(r) = h \tag{1}$$

and

$$r^2\phi' = c \qquad (' = d/dt), \tag{2}$$

the former representing the conservation of energy and the latter the conservation of angular momentum. Since ϕ is an ignorable coördinate, (1) may be replaced by

$$\tfrac{1}{2}r'^2 - U(r) + \tfrac{1}{2}c^2/r^2 = h \tag{1 bis}$$

i. e., by $\frac{1}{2}r'^2 - U^* = h$, where $U^* = U^*(r; c) = U(r) - \frac{1}{2}c^2/r^2$, so that

$$\tfrac{1}{2}r'^2 = U(r) - \tfrac{1}{2}c^2r^2 + h = F(r), \text{ say.}$$

We proceed to deduce from (1 bis) and (2) the nature of the motions corresponding to various forms of $U(r)$. Since the number of possibilities is rather large, it has been thought desirable to summarize the various cases and subcases in the following table:

I. $c = 0$. (A) $U = r^{-\lambda}$, $\qquad 0 < \lambda < +\infty$

(a) $h > 0$.

1. λ irrational. 2. λ rational $= p/q$.

1*. p odd, 2*. p even $= 2\kappa$.

2_1*. $\kappa + q$ odd, 2_2*. $\kappa +$

(b) $h = 0$.

(c) $h < 0$.

(B) $U = -\log r$

(C) $U = -r^{\lambda}$

(α) $0 < \lambda < 1$. (β) $1 \leqq \lambda < +\infty$.

(a) $h > 0$. (a) $h > 0$.

(b) $h \leqq 0$. (b) $h \leqq 0$.

II. $c \neq 0$. $(c > 0)$.

(A) $U = r^{-\lambda}$.

(α) $0 < \lambda < 2$.

(a) $h > 0$. (b) $h = 0$. (c) $h < 0$.

(i) $F(r_0) > 0$, (ii) $F(r_0) = 0$, (iii) $F(r_0) < 0$.

(β) $\lambda = 2$. $1 - \frac{1}{2}c^2 = \zeta$.

(aa) $\zeta > 0, h > 0$; (ab) $\zeta > 0, h < 0$; (ac) $\zeta > 0, h = 0$.

(ba) $\zeta = 0, h > 0$; (bb) $\zeta =, 0, h = 0$, (bc) $\zeta = 0, h < 0$.

(c) $\zeta < 0$.

(γ) $2 < \lambda < +\infty$.

(a) $h > 0$.

(i) $F(r_0) < 0$. (i_1) (i_2)

(ii) $F(r_0) = 0$. } 1. λ irrational, 2. λ rational $= p/q$.

(iii) $F(r_0) > 0$.

1*. p odd.

2*. p even $= 2\kappa$.

2_1*. $\kappa + q$ odd.

2_2*. $\kappa + q$ even.

(b) $h \leqq 0$.

(B) $U = -\log r$.

(i) $h < \log c + \frac{1}{2}$, (ii) $h = \log c + \frac{1}{2}$, (iii) $h > \log c + \frac{1}{2}$.

(C) $U = -r^{\lambda}$, $0 < \lambda < +\infty$.

(i) $F(r_0) < 0$, (ii) $F(r_0) = 0$, (iii) $F(r_0) > 0$.

2. The case of vanishing angular momentum. For convenience in treating the individual cases, we have written (1 bis) in the form

$$\frac{1}{2}r'^2 = U(r) - \frac{1}{2}c^2/r^2 + h = F(r, h) = F(r) \tag{3}$$

whence

$$2^{\frac{1}{2}}(t - t_0) = \int_{r_0}^{r} \pm |F(r, h)|^{-\frac{1}{2}} dr, \qquad (t_0 = \text{arb. const.}). \tag{3 bis}$$

We shall first suppose that

(I) $c=0$; (A) $U=r^{-\lambda}$, $(0<\lambda<+\infty)$; and (a) $h>0$.

Under the assumption (I), the motion is rectilinear, so that we may write $x\ (>0)$ for r, and (3) then becomes

(3 bis) $$\tfrac{1}{2}x'^2 = x^{-\lambda}+h = F(x),$$

whence

$$2^{\frac{1}{2}}t = \int_0^x (x^{-\lambda}+h)^{-\frac{1}{2}}dx = \int_0^x x^{\lambda/2}(1+hx^{\lambda})^{-\frac{1}{2}}dx.$$

As $x\to 0$, $t\to 0$, so that a collision occurs, corresponding to $t=0$. To investigate its nature, we consider only values of x for which $0<hx^{\lambda}<\rho$, say, $(\rho<1)$. Then the parenthesis in the preceding integral may be expanded by the binomial theorem and the resulting series integrated term by term, leading to the equation

$$2^{\frac{1}{2}}t = x^{\frac{1}{2}(\lambda+2)}\{2/(\lambda+2) - h/(3\lambda+2)x^{\lambda} + 3h^2/[4(5\lambda+2)]x^{2\lambda} - 5h^3/[8(7\lambda+2)]x^{3\lambda}+\cdots\}.$$

Regarding now t and x as complex variables, we set $x^{\lambda}=y$. Then

$$z = t^{2\lambda/(\lambda+2)} = 2^{-\lambda/(\lambda+2)}y\{2/(\lambda+2) - h/(3\lambda+2)y + 3h^2/[4(5\lambda+2)]y^2 - 5h^3/[8(7\lambda+2)]y^3+\cdots\}^{2\lambda/(\lambda+2)},$$

$x^{\lambda}, t^{2\lambda/(\lambda+2)}, \cdots$ denoting those branches which are real and positive for real positive values of the argument. Thus $z=y\mathfrak{A}(y)$, where $\mathfrak{A}(y)$ is a regular analytic function of y in some neighborhood of $y=0$ and $\mathfrak{A}(0)\neq 0$. Consequently, $z'(0)\neq 0$, so that y is a regular analytic function of z in some neighborhood of $z=0$ and may be developed into a power series there, and since the coefficients of the series for z are real, the same is true of the series for y. On expressing y, z in terms of x, t respectively, this leads to

(4) $x=\beta_1 t^{2/(\lambda+2)}(1+\beta_2 t^{\nu}+\beta_3 t^{2\nu}+\cdots)$, where $\beta_1>0$ and $\nu=2\lambda(\lambda+2)$,

the Puiseux series in parenthesis converging uniformly for all t such that $|t|<\rho_1$, say, $\rho_1>0$.

Two cases arise:

1. λ is irrational. Then $2/(\lambda+2)$ is also irrational, and x has a logarithmic branch point at $t=0$, so that $x(t)$ possesses no analytic continuation through $t=0$.

To prove this, we observe that, if we start with that branch of x which is real and positive for real positive values of t (which is seen to be possible from the fact that the coefficients in the expression for x are real and $\beta_1>0$),

then since $2/(\lambda+2)$ is irrational, it is clear that no number of circuits about $t=0$ will restore the first factor on the right of (4) to a real value. If, then, x is to assume its original value, it must, in particular, be real and this is possible only if, after n circuits, the second factor is proportional to the conjugate of the first, with a real proportionality factor k. Put $t=\rho\exp(i\vartheta)$. Then after n circuits

$$t^{2/(\lambda+2)}=\rho^{2/(\lambda+2)}\exp(i4\pi n/(\lambda+2))=\rho^{2/(\lambda+2)}\exp(i2\pi\alpha),$$

where $\alpha=(2n/(\lambda+2))=2n/(\lambda+2)-[2n/(\lambda+2)]$.

The second factor on the right-hand side of (4) would then have to be of the form $k\rho^{2/(\lambda+2)}\exp(-i2\pi\alpha)$ for all $\rho<\rho_1$. But this is impossible, since inspection of the factor in question shows that, for ρ sufficiently small, it lies in an arbitrarily small neighborhood of $t=1$ and hence in the interior of the angle bounded by the half-rays $\vartheta=\pm 2\pi\alpha$. From this contradiction, it follows that x has a logarithmic branch point at $t=0$.

2. λ is rational, say of form p/q, where p and q are relatively prime. Reasoning similar to that of case 1 shows that x is real if and only if $t^{2/(\lambda+2)}$ is real. Therefore, x will be continuable through $t=0$ under and only under conditions such that $t^{2/(\lambda+2)}$ is real for negative t.

This last factor may be written as $t^{2q/(p+2q)}$, where $2q/(p+2q)$ is to be interpreted as reduced to its lowest terms, viz., $2q/(p+2q)$, if p is odd; $q/(\frac{1}{2}p+q)$, if p is even. Then

(1*) If p is odd, $p+2q$ is odd and real analytic continuation is always possible, q arbitrary. This includes the Newtonian case $p=q=1$.

(2*) If p is even, put $p=2\kappa$. Then $2/(\lambda+2)$ becomes $q/(\kappa+q)$, and

(2_1*) If $\kappa+q$ is odd, real analytic continuation is again possible;

(2_2*) If $\kappa+q$ is even, q must be odd, since otherwise we should not have p and q relatively prime. Then κ is necessarily odd. In this case, real analytic continuation is impossible, e. g. $p=2$, $q=1$.

If the positive square root be chosen in (3 bis), it is seen that x increases steadily with t. As $x\to\infty$, $t\to\infty$ and $x\sim(2h)^{\frac{1}{2}}t$. Hence the solution is not bounded. If there are in general to be equilibrium solutions or asymptotic solutions, we must have [2] for some r_0

$$F(r_0)=F_r(r_0)=0, \tag{5}$$

[2] Cf. equation (3) above and A. Wintner, *Analytical Foundations of Celestial Mechanics*, p. 132 and p. 153.

or in the present case, $F = F_x = 0$. But from (3 bis) it is seen that neither F nor F_x vanishes for $0 < x < +\infty$, so that under the present assumptions neither equilibrium solutions nor asymptotic solutions exist.

Suppose next that

(b) $h = 0$. Here the motion is the same as in (a), with respect both to boundedness and to the nature of the collision and to the non-existence of equilibrium or asymptotic solutions, but the results are more easily obtained, since

$$t = 2^{-\frac{1}{2}} \int_0^x x^{\frac{1}{2}\lambda} dx = 2^{\frac{1}{2}} x^{\frac{1}{2}(\lambda+2)}/(\lambda+2) \text{ or } x = \text{const. } t^{2/(\lambda+2)}.$$

Again, suppose that

(c) $h < 0$. Then the nature of the collision is as before and there are no equilibrium solutions or asymptotic paths, since $F_x \neq 0$, $(0 < x < +\infty)$, but $x \sim (2h)^{\frac{1}{2}}t$ implies that the solution is bounded.

Let assumption (I) be retained, but (A) be replaced by

$$U = -\log r, \qquad (r = x), \tag{B}$$

so that $\frac{1}{2}x'^2 = F(x) = -\log x + h$; $F_x = -x^{-1}$; $F_{xx} = x^{-2}$. Thus there are no finite positive zeros of either F_x or F_{xx}. For all $h \gtreqless 0$, $F(0+) = +\infty$, $F(+\infty) = -\infty$. The path is rectilinear and has a maximum at $x = e^h$ corresponding to a cusp.[3]

Since

$$dt = 2^{-\frac{1}{2}}(h - \log x)^{-\frac{1}{2}} dx, \tag{6}$$

it follows that, if t^* be the time arbitrarily corresponding to the position x^*, $(0 < x^* < e^h)$, then

$$t - t^* = 2^{-\frac{1}{2}} \int_{x^*}^x (h - \log x)^{-\frac{1}{2}} dx, \tag{7}$$

from which it is seen that the solution path, starting from x^*, reaches both $x = 0$ and $x = e^h$ in finite time. To $x = 0$ may be made to correspond $t = 0$, these representing the place and the time of collision.

A necessary condition that $x(t)$ be continuable analytically through $t = 0$ is that it should have at most an algebraic singularity there, whereupon the same would be true of the inverse function $t = t(x)$ and hence of the derivative of the latter, since differentiation cannot introduce transcendental singularities. But (6) shows that this derivative has a transcendental (loga-

[3] Cf. A. Wintner, *op. cit.*, §§ 185, 186.

rithmic) singularity at $x = 0$, from which it follows that the analytic continuation of $x = x(t)$ through $t = 0$ is impossible.[4]

As in the preceding case, there are no equilibrium solutions and no asymptotic paths.

Continuing the assumption $c = 0$, we now suppose

(C) $$U = U(x) = -x^\lambda,$$

(α) $$0 < \lambda < 1,$$

then $F(x) = -x^\lambda + h$; $F_x = -\lambda x^{\lambda-1}$; $F_{xx} = -\lambda(\lambda-1)x^{\lambda-2}$, so that F_x and F_{xx} are $\neq 0$ for all x, $0 < x < +\infty$. Suppose further, that

(a) $$h > 0.$$

Then $F(0) = h > 0$, $F(+\infty) = -\infty$, so that there exists a bounded collision path, the maximum value of x corresponding to a cusp. As before, there are no equilibrium solutions or asymptotic solutions. To examine the nature of the collision, we consider

(8) $$x'^2 = 2(h - x^\lambda).$$

Taking, as we may, the positive square root $x' = (2h)^{\frac{1}{2}}(1 - x^\lambda/h)^{\frac{1}{2}}$, and setting $x/h^{1/\lambda} = r$, we may expand the parenthesis by the binomial theorem and the resulting series converges uniformly for $|r^\lambda| < \rho$, say, $(\rho < 1)$, and may be integrated there term by term, leading to the expression

$$t = kr\left(1 + \frac{1}{2(\lambda+1)}r^\lambda + \frac{3}{8(2\lambda+1)}r^{2\lambda} + \frac{5}{16(3\lambda+1)}r^{3\lambda} + \cdots\right), \qquad (k = \text{const.} \neq 0).$$

Regarding t and r as complex variables, we put $r^\lambda = y$, so that

$$z = t^\lambda = k^\lambda y\left(1 + \frac{1}{2(\lambda+1)}y + \frac{3}{8(2\lambda+1)}y^2 + \cdots\right)^\lambda,$$

$z = t^\lambda$ and $y = r^\lambda$ denoting those branches which are real for real values of the argument. Then, as in case (Aa), $z = y\mathfrak{A}(y)$, where $\mathfrak{A}(y)$ is a regular

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[4] This fact may also be verified by making the substitution $\log x = y$, followed by $y = h - z^2$, whereupon (7) becomes $t - t^* = -\sqrt{2}\exp(h)\int_{z^*}^{z}\exp(-z^2)dz$. If $z = z(t)$ represents the inverse of the foregoing, then x is given as a function of t by $x = \exp(h - z^2(t))$. Since the nature of the transcendental singularity of $t = t(z)$ at $z = \infty$ is quite different from that of $y = \log x$ at $x = 0$ (or $x = \infty$), it follows that x as a function of t has a transcendental singularity at $t = 0$. For the nature of $t(z)$ at $z = \infty$, cf. Polya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin (1925), vol. II, p. 33, no. 189.

analytic function of y in some neighborhood of $y = 0$ and $\mathfrak{A}(0) \neq 0$, so that y may be expressed as a regular analytic function of z in some neighborhood of $z = 0$, and this leads to an expression of the form

$$x = \gamma_1 t[1 + \gamma_k t^{(k-1)\lambda}(1 + \gamma_{k+1} t^{\lambda} + \gamma_{k+2} t^{2\lambda} + \cdots)],$$

where $\gamma_1 > 0$ and $\gamma_k \neq 0$, $(k \geqq 2)$, whereupon reasoning similar to that of case (Aa), shows that

1. If λ is irrational, no analytic continuation is possible.
2. If λ is rational, $= p/q$ say, where p and q are relatively prime, we write $(k-1)p/q = m/s$, where it is understood that m and s are relatively prime. Here real analytic continuation is possible, if s is odd, but not if s is even.

If, on the other hand,

(b) $$h \leqq 0,$$

it is clear from (8) and from the expression for F_x that there is no motion at all.

Finally, if

(β) $$1 \leqq \lambda < +\infty$$

and (a) : $h > 0$, there exists, as before, a bounded collision path, the situation with respect to the behavior at the maximum value of x and with respect to the analytic nature of the collision being precisely as in case (α). In the cases in which analytic continuation through the origin is possible and $\lambda > 1$, the motion is similar to the case $\lambda = 2$ of the harmonic oscillator. Since F and F_x do not vanish simultaneously, there are no equilibrium or asymptotic solutions.

If, however, (b) : $h \leqq 0$, then there is no motion except, in the case $h = 0$, for the trivial equilibrium solution $x \equiv 0$.

3. The case of non-vanishing angular momentum. Having considered linear motions, we now turn to the case

(II) $$c \neq 0,$$

where we may suppose without restriction that $c > 0$. As our first sub-case, we shall consider that in which

(A) $$U(r) = r^{-\lambda} \text{ and } (\alpha)\ 0 < \lambda < 2.$$

Here (3) becomes

(9) $$\tfrac{1}{2} r'^2 = r^{-\lambda} - \tfrac{1}{2} c^2 r^{-2} + h = F(r),$$

whence

(10) $F_r(r) = -\lambda r^{-\lambda-1} + c^2 r^{-3}$ and (11) $F_{rr}(r) = \lambda(\lambda+1) r^{-\lambda-2} - 3c^2 r^{-4}$.

As $r \to 0+$, $r'^2 \sim -c^2/r^2$, which is impossible for real r, so that there is no collision in any of the three cases $h \gtreqless 0$.

As $r \to +\infty$, $r'^2 \sim 2h$, so that if (a): $h > 0$, then $r \sim kt$, k a constant, which may be supposed positive. In this case, a solution exists and is not bounded. dr/dt has just one zero, since F_r has but one positive real zero and F crosses the r-axis an odd number of times, therefore once. The zero of dr/dt corresponds to a minimum value of r, which we may suppose to occur at $t = 0$. For the nature of the path at this point, cf. the reference under (i) below. By (2), $\phi' = c/r^2$, $(c > 0)$, so that ϕ increases steadily with t. Moreover, as $r \sim kt$, integration of (2) shows that ϕ approaches a finite limit as $t \to +\infty$, and a similar result holds if $t \to -\infty$.

In this case, $F(r_0) = F_r(r_0) = 0$ would imply $F_{rr}(r_0) = 0$ also. But the latter two equations imply $\lambda = 2$, contrary to hypothesis, so that no equilibrium or asymptotic solutions exist.

Again, if (b): $h = 0$, $2(dt/dr)^2 \sim r^\lambda$, so that $r \sim \text{const.}\, t^{2/(\lambda+2)}$. Here r becomes infinite more slowly than in case (a), but the motion is otherwise the same.

Thirdly, if (c): $h < 0$, any possible solution will be bounded. To determine whether or not a solution exists at all, we observe from (11) that

$$F_{rr}(r) \gtreqless 0, \text{ according as } r \lesseqgtr r_1 = \left(\frac{3c^2}{\lambda(\lambda+1)}\right)^{1/(2-\lambda)}.$$

Consequently, $F(r)$ is first concave and then convex from below. The extremum of $F(r)$ is located at $r_0 = (c^2/\lambda)^{1/(2-\lambda)}$, where $F_r(r_0) = 0$. For $0 < \lambda < 2$, it is seen that $r_0 < r_1$, so that the extremum is a maximum. $F(0+) = -\infty$, $F(+\infty) < 0$. Therefore,

(i) if $F(r_0) > 0$, there will be a bounded (periodic) non-circular motion. for a more detailed discussion of which the reader is referred to A. Wintner, *The Analytical Foundations of Celestial Mechanics*, § 215, p. 153;

(ii) if $F(r_0) = 0$, there will be a circular motion and nothing else;

(iii) if $F(r_0) < 0$, no solution will exist.

In case (i), $0 < R_0 \leqq r \leqq R_1$, say, so that from (2) $|\phi - \phi_0| \geqq cR_1^{-2}|t|$, and ϕ becomes positively or negatively infinite with t, and the same is true in case (ii).

F and F_r vanish simultaneously in, and only in, case (ii). The circular path then represents an equilibrium solution, but there are no asymptotic paths.

Under the assumptions (II) : $c > 0$ and (A) : $U = r^{-\lambda}$, we now consider case

(β) $$\lambda = 2.$$

We set $1 - \frac{1}{2}c^2 = \zeta$, and examine first the subcase corresponding to (aa) : $\zeta > 0$; $h > 0$. Then

(12) $r'^2 = 2(h + \zeta/r^2)$, whence (13) $r = 2^{\frac{1}{3}}t^{\frac{1}{3}}((2\zeta)^{\frac{1}{3}} + ht)^{\frac{1}{3}}$,

if $t = 0$ corresponds to $r = 0$, and conversely. In this case, there is a collision, but no real analytic continuation of r through $t = 0$ is possible. Substitution of (13) into (2) and subsequent integration leads to

$$\phi - \phi_0 = c[2(2\zeta)^{\frac{1}{3}}]^{-1}\log[t/((2\zeta)^{\frac{1}{3}} + ht)],$$

so that ϕ becomes logarithmically infinite as $t \to 0+$. As t increases and becomes positively infinite, the same is true of r. ϕ also increases steadily, but approaches a finite limit as $t \to +\infty$.

If, on the other hand, (ab) : $h < 0$, the situation with respect to the collision as $t \to 0+$ is the same as before, but as t increases, r remains bounded. ϕ increases steadily with t and becomes infinite as $t \to -(2\zeta)^{\frac{1}{3}}/h > 0$, while as $t \to 0+$, ϕ becomes negatively infinite.

In both the foregoing cases, $x = r\cos\phi$ becomes

$$x = 2^{\frac{1}{3}}t^{\frac{1}{3}}((2\zeta)^{\frac{1}{3}} + ht)^{\frac{1}{3}}\cos\{c(8\zeta)^{-\frac{1}{2}}\log[t/((2\zeta)^{\frac{1}{3}} + ht)]\},$$

and the expression for $y = r\sin\phi$ is similar.

Thirdly, if (ac) : $h = 0$, we have $r'^2 = 2\zeta/r^2$, so $r = (8\zeta)^{\frac{1}{4}}t^{\frac{1}{2}}$. Here, as before, there is a collision, and real analytic continuation of r through $t = 0$ is impossible. $\phi' = c/r^2 = c/((8\zeta)^{\frac{1}{2}}t)$; $\phi = c(8\zeta)^{-\frac{1}{2}}\log t + \text{const.}$, so that $\phi \to -\infty$ as $t \to 0+$; $\phi \to +\infty$ as $t \to +\infty$, in which latter case $r \to +\infty$ also.

Again, if (ba) : $\zeta = 0$, $h > 0$, (12) reduces to $r'^2 = 2h$, so $r = (2h)^{\frac{1}{2}}t$. In this case, r can be continued analytically through $t = 0$, $\phi' = c/2ht^2$; $\phi = -c(2h)^{-1}t^{-1}$, so that as $t \to 0+$, ϕ becomes negatively infinite (assuming $c > 0$), but more strongly than in the preceding three cases. As $t \to +\infty$, $r \to +\infty$ and $\phi \to 0-$. $x = (2h)^{\frac{1}{2}}t\cos(c/2ht)$, $y = -(2h)^{\frac{1}{2}}t\sin(c/2ht)$.

If (bb) : $\zeta = 0$, $h = 0$, (12) reduces to $r' = 0$, $r = \rho = \text{const.}$ and $\phi = c\rho^{-2}t + \text{const.}$, where the latter constant may be taken to be zero, and we have circular motion.

If (bc) : $\zeta = 0$, $h < 0$, there is no real solution.

Finally if (c) : $\zeta < 0$, there is no collision for any value of h. If $h \leqq 0$, there is no real solution at all, just as in case (bc). If, on the other hand, $h > 0$,

$$t = 2^{-\frac{1}{2}} \int_R^r r(hr^2 + \zeta)^{-\frac{1}{2}} dr = (hr^2 + \zeta)^{\frac{1}{2}}/h2^{\frac{1}{2}},$$

where R is the minimum value of r, corresponding to $t = 0$. For the behavior of the path there, cf. the reference under case (II A α c i). $r^2 = h^{-1}(2h^2t^2 - \zeta)$, so that upon substitution in (2). and integration, $\phi = c\eta/(2h)$ arctan (ηt), where $\eta^2 = 2h^2/(-\zeta)$, from which it appears that ϕ approaches a finite limit as $t \to 0+$ and as $t \to +\infty$, while in the latter case, $r \to +\infty$.

The usual tests applied to the case $\lambda = 2$ show that there are no equilibrium solutions except in the case (bb), where $F \equiv 0$ and $F_r \equiv 0$, in which case all circular motions are equilibrium solutions. In no case do there exist paths asymptotic as $t \to \infty$.

We next assume (II) : $c > 0$; (A) : $U = r^{-\lambda}$;

$$(\gamma) \qquad 2 < \lambda < +\infty.$$

In this case, $r_1 < r_0$ (cf. p. 485 for the notation), so that the extremum of $F(r)$ is a minimum.

We first suppose (a) : $h > 0$, so that it is seen from (9) that $F(r)$ tends to $+\infty$ as $r \to 0+$ and is positive as $r \to +\infty$. Then according as

$$\text{(i)} \quad F(r_0) < 0, \qquad \text{(ii)} \quad F(r_0) = 0, \qquad \text{(iii)} \quad F(r_0) > 0,$$

there will be

(i_1) a bounded collision path *and* (i_2) an unbounded path without collision;

(ii) a circular equilibrium path and asymptotic paths corresponding to (i_1) and (i_2);

(iii) an unbounded collision path.

We examine first case (i_1). Here

$$\tfrac{1}{2}r'^2 = r^{-\lambda} - \tfrac{1}{2}c^2r^{-2} + h, \qquad (2 < \lambda < +\infty),$$

so that

$$(14) \quad t = 2^{-\frac{1}{2}} \int_0^r r^{\lambda/2}\{1 + (hr^\lambda - \tfrac{1}{2}c^2r^{\lambda-2})\}^{-\frac{1}{2}} dr,$$

$$(15) \quad t = 2^{-\frac{1}{2}} \int_0^r r^{\lambda/2}\{1 - \tfrac{1}{2}(hr^\lambda - \tfrac{1}{2}c^2r^{\lambda-2}) + \tfrac{3}{8}(hr^\lambda - \tfrac{1}{2}c^2r^{\lambda-2})^2 - \cdots\} dr.$$

The series in brackets converges uniformly and absolutely for $|r^{\lambda-2}|(\frac{1}{2}c^2 + |hr^2|) < \rho$, say, $(0 < \rho < 1)$, i. e. for $|r| < r^{**}$, for a suitable $r^{**} > 0$

Consequently, (15) may be integrated term by term there and the resulting series will converge uniformly for r in the above range. Before carrying out this integration, it is convenient to write (15) as

$$t = 2^{-\frac{1}{2}} \int_0^r r^{\lambda/2}\{1 - \tfrac{1}{2}r^{\lambda-2}(hr^2 - \tfrac{1}{2}c^2) + \tfrac{3}{8}r^{2(\lambda-2)}(hr^2 - \tfrac{1}{2}c^2)^2 + \cdots\}dr$$

$$= 2^{-\frac{1}{2}} \int_0^r \{r^{\lambda/2} - \tfrac{1}{2}hr^{3\lambda/2} + \tfrac{1}{4}c^2r^{3\lambda/2-2} + \tfrac{3}{8}h^2r^{5\lambda/2} - \tfrac{3}{8}hc^2r^{5\lambda/2-2} + \cdots\}dr,$$

so that

$$(16)\quad t = r^{\frac{1}{2}(\lambda+2)}\left\{\frac{2^{\frac{1}{2}}}{\lambda+2} - \frac{h}{2^{\frac{1}{2}}(3\lambda+2)}r^{\lambda} + \frac{c^2}{2^{3/2}(3\lambda-2)}r^{\lambda-2} + \frac{3h^2}{2^{5/2}(5\lambda+2)}r^{2\lambda} - \frac{3hc^2}{2^{5/2}(5\lambda-2)}r^{2\lambda-2} + \cdots\right\},$$

i. e., t is of the form

$$(17)\qquad t = r^{\frac{1}{2}(\lambda+2)}\{a_1 + r^{\lambda-2}[a_2 + a_3r^{\mu_3} + a_4r^{\mu_4} + \cdots]\}$$

or

$$(18)\qquad t = r^{\frac{1}{2}(\lambda+2)}\{a_1 + a_2r^{\nu_2} + a_3r^{\nu_3} + a_4r^{\nu_4} + \cdots\},$$

where t and r are henceforth to be regarded as complex variables, the coefficients and the exponents, however, being real and r^ν for any exponent ν referring to that branch of the function which is real and positive for positive real r.

From (16) and the remarks preceding it, it is seen that t is a regular analytic function of r in the domain $0 < |r| < r^{**}$. Moreover, from the uniform convergence of the integrated series, it follows by the approximation theorem for almost-periodic functions of a complex variable[5] that t is an almost-periodic function of r in that domain. Furthermore, if λ is irrational, t has an almost-periodic singularity (corresponding to a logarithmic branch point) at $r = 0$, as is seen by applying to (17), where $a_1 > 0$, the reasoning used in case (I A a 1). Also, since $\lambda > 2$, (16) shows that the exponents of the series for t are all positive and that amongst them there is a smallest, viz., $\frac{1}{2}(\lambda+2)$. Consequently, $t = f(r)$ is an analytic almost-periodic function in the domain $0 < |r| < r^{**}$ and has a normal almost-periodic singularity at the point $r = 0$. Then, by a theorem of Bohr,[6] for a sufficiently small $r^* > 0$ the function $t = f(r)$ will represent the neighborhood E_{r^*} of $r = 0$ on a simply connected domain D of the Riemann surface S_t, which domain D

[5] Cf. H. Bohr, "On the inverse function of an analytic almost periodic function," *Annals of Mathematics*, 2nd Ser., vol. 32 (1931), p. 249.

[6] Cf. H. Bohr, *ibid.*, p. 251.

will contain a complete neighborhood E_{ρ^*} of $t = 0$. Hence in E_{ρ^*} there exists an inverse function $r = g(t)$ and this function will (for sufficiently small ρ^*) be analytic and almost-periodic in $0 < |t| < \rho^*$ and have a normal almost-periodic singularity at $t = 0$.

Furthermore, if $\Lambda = \frac{1}{2}(\lambda + 2) > 0$ denotes the smallest exponent of $f(r)$ and $M > 0$ the smallest exponent of $g(t)$, then

$$\mathrm{M} = 1/\Lambda = 2/(\lambda + 2),$$

and if we write $f(r) = r^{\Lambda} f_1(r)$ and $g(t) = t^{\mathrm{M}} g_1(t)$, the modules of $f_1(r)$ and of $g_1(t)$ will be connected by the equation

$$\mathfrak{M} g_1 = \Lambda^{-1} \mathfrak{M} f_1,$$

(where it is seen from (16) that $\mathfrak{M} f_1$ is generated by λ and 1). Consequently, the reasoning of case (I A a 1), applied above to (17), now shows that r as a function of t possesses, in the case λ irrational, a logarithmic branch point at $t = 0$, so that in this case analytic continuation of the motion through the instant of collision is impossible.

If, on the other hand, λ is rational, the right-hand member of (16) becomes a Puiseux series which may be inverted just as in case (I A a). As there, we conclude that if $\lambda = p/q$, p and q relatively prime, then

1* If p is odd, q arbitrary, real analytic continuation is possible;

2* If p is even, $= 2\kappa$, then
(2_1*) If $\kappa + q$ is odd, real analytic continuation is again possible.
(2_2*) If $\kappa + q$ is even, no real analytic continuation is possible.

The smaller zero of $F(r)$ is simple and corresponds to a maximum R_1 of $r = r(t)$, which, by (3 bis), is reached in finite time.[3] r decreases steadily with t from $r = R_1$ to $r = 0$, the point of collision. From (2) it follows that ϕ increases steadily with t as r increases from 0 to R_1 and then decreases to 0. If $0 < r^* < R_1$ and if t^*, t_1 be the times corresponding to r^*, R_1 respectively, then the finite value of $\phi - \phi^* = \int_{t^*}^{t} (c/r^2)\,dt$ for all t, $t^* \leqq t \leqq t_1$ shows that ϕ approaches a finite limit as $r \to R_1$. If $t \to 0$, then $r \to 0$ and

$$\phi - \phi_0 \sim k \int_0^t t^{-4/(\lambda+2)}\,dt, \qquad k = \text{const.}, \ \lambda > 2,$$

so that ϕ approaches a finite limit in this case also.

With respect to the case (II A γ ai$_2$), it is seen that R_2, the larger zero of $F(r)$, is also simple and corresponds to a minimum of $r = r(t)$, which is

reached in a finite time from a point on the path for which $r = r^{**} > R_2$. As t varies in any finite range, the same is true of ϕ, so that ϕ approaches a finite limit as $t \to 0$, if $t = 0$ is made to correspond to $r = R_2$. As $r \to +\infty$, $r' \sim (2h)^{\frac{1}{2}}$, so that $r \sim (2h)^{\frac{1}{2}}t$ and by (2) $\phi \sim k_1 t^{-1} + k_2$, ($k_1$ and k_2 constants), so that ϕ approaches a finite limit as $t \to +\infty$ also. The motion is similar to that in cases (II A α a) and (II A α b). This completes the discussion of the motion under the assumptions (II): $c > 0$; (A): $U = r^{-\lambda}$; (γ): $2 < \lambda < +\infty$; (a) $h > 0$; (i) $F(r_0) < 0$.

If (ii): $F(r_0) = 0$, then, as already pointed out, there is a circular motion, corresponding to an equilibrium solution and asymptotic paths one lying inside the equilibrium solution and corresponding to case (i_1) above, while the other lies outside and corresponds to (i_2). The former is a collision path behaving at the moment of collision as the path in case (i_1) above. r increases steadily with t and approaches r_0 as $t \to +\infty$. ϕ likewise increases steadily with t and becomes infinite as $r \to r_0$ (i. e., as $t \to \infty$) as may readily be seen from the integration of (2) in view of the fact that r remains bounded. On the outer asymptotic path, t becomes infinite and ϕ approaches a finite limit as r, starting from some $r^{**} > r_0$, becomes infinite, just as in case (i_2), but as r decreases toward r_0, t becomes negatively infinite and ϕ decreases indefinitely with t.

If, finally (iii): $F(r_0) > 0$, then $F(r)$ has no zeros in $0 < r < +\infty$, so that there exists an unbounded collision path. The nature of the collision, corresponding to $t = 0$, is exactly the same as in subcase (i_1) above. As t increases from 0 to $+\infty$, r increases steadily and $\sim$ const. t, while ϕ approaches a finite limit, this portion of the path being similar to that of case (i_2).

Let assumptions (II), (A), (γ) be retained, but let (a) be replaced by (b): $h \leqq 0$. Here $F(0+) = +\infty$, but $F(r) < 0$ for all sufficiently large r. $F(r)$ crosses the r-axis an odd number of times, hence just once, as $F_r(r)$ has but one positive real zero. The motion corresponds to a bounded collision path precisely similar to that of subcase (i_1) under (a). This concludes the treatment of case (II A). The only equilibrium solutions or asymptotic paths are those treated under subcase (aii).

Retaining the assumption (II): $c > 0$, we now suppose that

(B) $$U = -\log r.$$

Then

$$F(r) = -\log r - \tfrac{1}{2}c^2 r^{-2} + h;\ F_r(r) = -r^{-1} + c^2 r^{-3};\ F_{rr}(r) = r^{-2} - 3c^2 r^{-4}.$$

$F_r = 0$ implies $r = c\ (>0)$, while $F_{rr} \gtreqless 0$ according as $r^2 \gtreqless 3c^2$. For all $h \gtreqless 0$, we have $F(0+) = F(+\infty) = -\infty$. $F(r)$ has one maximum, viz., at $r = c$, and $F(c) \gtreqless 0$ according as $h \gtreqless \log c + \frac{1}{2}$.

Consequently, (i) if $h < \log c + \frac{1}{2}$, there is no solution;

(ii) if $h = \log c + \frac{1}{2}$, there is a circular motion and nothing else;

(iii) if $h > \log c + \frac{1}{2}$, there is no collision, but a bounded periodic motion like that of case (II A α ci).

Finally let assumption (II) be retained but (B) be replaced by

(C) $$U = -r^\lambda, \qquad 0 < \lambda < +\infty$$

Then

$$F(r) = -r^\lambda - \tfrac{1}{2}c^2 r^{-2} + h;\ F_r(r) = -\lambda r^{\lambda-1} + c^2 r^{-3};\ F_{rr} = -\lambda(\lambda-1)r^{\lambda-2} - 3c^2 r^{-4}.$$

As in the previous case, for all h, $F(0+) = F(+\infty) = -\infty$. F_r has just one zero, viz., at $r_0 = (c^2/r)^{1/(\lambda+2)}$. Then

(α): if $0 < \lambda < 1$, it follows that $F_{rr} \gtreqless 0$ according as $r \gtreqless r_1 = (3c^2\lambda^{-1}(1-\lambda)^{-1})^{1/(\lambda+2)}$, respectively, and $r_0 < r_1$, since $1 < 3/(1-\lambda)$.

(β): if $\lambda = 1$, direct calculation shows that F_r vanishes at $r_0 = c^{2/3}$, while $F_{rr} = -3c^2 r^4 < 0$.

(γ) if $\lambda > 1$, F_r vanishes at $r_0 = (c^2/\lambda)^{1/(\lambda+2)}$ and $F_{rr} < 0$ for all $r > 0$.

Hence the extremum is a maximum, and according as $F(r_0) \gtreqless 0$, there exists (i) no solution; (ii) a circular motion and nothing else; (iii) a bounded periodic motion like that of case (II A α ci).

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RAMANUJAN IDENTITIES INVOLVING THE PARTITION FUNCTION FOR THE MODULI 11^α.*

By JOSEPH LEHNER.

1. Introduction.[1] Ramanujan conjectured that

$$p(n) \equiv 0 \pmod{q} \quad \text{if} \quad 24n \equiv 1 \pmod{q}, \tag{1.1}$$

where $p(n)$, with $n > 0$, is the number of unrestricted partitions of n, and q is one of the primes 5, 7, 11. Direct proofs of each of these three congruences were given by Ramanujan.[2]

He also indicated an indirect method of proof; namely, as immediate consequences of certain *identities*, which he stated without proof, and of which the following is an example:

$$\sum_{n=0}^{\infty} p(7n+5)x^n = 7\,\frac{\prod_1^\infty (1-x^{7m})^3}{\prod_1^\infty (1-x^m)^4} + 49x\,\frac{\prod_1^\infty (1-x^{7m})^7}{\prod_1^\infty (1-x^m)^8}. \tag{1.2}$$

The identities for the moduli 5 and 7 have since been proved by several writers.[3] Ramanujan did not give an identity for $q = 11$.

Ramanujan stated more than (1.1). His most general conjecture—that

$$p(n) \equiv 0 \pmod{q^\alpha} \text{ if } 24n \equiv 1 \pmod{q^\alpha}, \quad \alpha = \text{integer} > 0 \tag{1.3}$$

—is known to be false for $q = 7$, $\alpha > 2$, but G. N. Watson[4] [15] has proved the correctness of (1.3) for $q = 5$, $\alpha \geqq 1$, and has proposed and proved a substitute congruence for $q = 7$ and all α. In the same paper he promised a discussion of the moduli 11^α, but to date it has not appeared. A proof of (1.3) for $q^\alpha = 11^2$ is contained in one of Ramanujan's notebooks but has not been published.[5] For higher powers of 11, nothing is known concerning the

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[1] The author wishes to express his thanks to Professor H. Rademacher, who suggested to him the problem treated in this paper, and to Professor J. B. Rosser for helpful advice in connection with the computations of 9.

[2] [14], pp. 210-213, 232-238. (Square brackets refer to the bibliography at the end of this paper.)

[3] For references see footnotes 3, 4, 4a in [11].

[4] See also [16] and the reference in footnote 3 of that paper.

[5] [4], p. 100.

correctness of (1.3), except for an affirmative test by D. H. Lehmer [6] of an isolated value.

The procedure of Watson and Zuckerman was to establish identities of type (1.2) containing a factor q^{α} in the right member. Recently Rademacher [12] developed a systematic method for the investigation of these identities, and applied it to the moduli 5^{α}, 7, 7^2, 13^{α}. Rademacher's method displays very clearly the underlying connection between the Ramanujan identities and the theory of modular functions. But it is not so powerful as Watson's process, which employs modular equations: it does not establish, except for the first few values of α where direct computation is possible, the *integrality* of the coefficients of the modular functions (which themselves have integral coefficient expansions) in the identity, and so the *congruence* (1.3) does not follow.

In the present paper we use Rademacher's method to establish identities for the moduli 11^{α}. For $\alpha = 1$, the coefficients in the identity are computed—**6**—while in the case $\alpha = 2$—**9**—they are determined sufficiently to prove the congruence (1.3). For higher values of α we cannot prove the integrality of the coefficients in the identities and so cannot prove (1.3). Thus, our goal, the proof or disproof of the congruences (1.3) in the unknown cases (i.e., $\alpha > 2$), is not reached.

It is felt that the identities possess some interest in themselves, particularly since they highlight the arithmetic character inherent in the larger part of the theory of modular functions. From this point of view the Ramanujan congruences appear to be an arithmetic expression of certain facts pertaining to special subgroups of the modular group and the functions belonging to them.

2. Outline of the method. We shall make use of a simple modification of an operator studied by Hecke.[7] Let $F(\tau)$ be an entire modular function belonging to the subgroup $\Gamma_0(q)$ of the full modular group. $\Gamma_0(q)$ is defined by the condition $c \equiv 0 \pmod{q}$ in the substitution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where q is a prime > 3. We define the linear operator

$$U_q F(\tau) = q^{-1} \sum_{\lambda=0}^{q-1} F\left(\frac{\tau + \lambda}{q}\right). \tag{2.1}$$

In general, we cannot prove that $U_q F$ is entire modular on $\Gamma_0(q)$. However, if F exhibits a simple behavior under the (non-modular) substitution $\tau' = -1/q\tau$, such as

$$F(-1/q\tau) = cF(\tau), \text{ or } F(-1/q\tau) = cF^{-1}(\tau),\ c = \text{const.}, \tag{2.2}$$

[6] See [9], p. 90.

[7] See [5] for a readable sketch of this theory and [6] for a complete account.

then $U_q F$ will belong to $\Gamma_0(q)$. Moreover, in certain cases F may be a modular function of $\Gamma_0(q^2)$, and not of $\Gamma_0(q)$, but still the same conclusion will hold. This is true of the function

$$\Phi(\tau; q^2) = \frac{\eta(q^2\tau)}{\eta(\tau)}, \tag{2.3}$$

where $\eta(\tau)$ is the well-known Dedekind function

$$\eta(\tau) = \exp(\pi i\tau/12) \cdot \prod_1^\infty (1 - x^m), \quad x = \exp 2\pi i\tau. \tag{2.4}$$

$\Phi(\tau; q^2)$ is an entire modular function on $\Gamma_0(q^2)$, but $U_q\Phi(\tau; q^2)$ belongs to the larger group $\Gamma_0(q)$.

We now take $q = 11$ and show in **5** that every entire modular function of $\Gamma_0(11)$ is a rational function of certain basic functions $A_{11}(\tau)$, $C_{11}(\tau)$. The latter have power series expansions in $x = \exp 2\pi i\tau$ with integral coefficients. We find in **6** that

$$U_{11}\Phi(\tau; 11^2) = 11(11AC^2 - 11^2C + 2AC - 32C - 2), \tag{2.5}$$
$$A = A_{11}(\tau), \quad C = C_{11}(\tau).$$

On the other hand, we have as a special case ($q = 11$, $\alpha = 1$) of Theorem 7,

$$U_{11}\Phi(\tau; 121) = x\prod_1^\infty (1 - x^m) \sum_{l=0}^\infty p(11l + 6)x^l, \quad x = \exp 2\pi i\tau. \tag{2.6}$$

This, together with (2.5), gives

$$\sum_{l=0}^\infty p(11l + 6)x^{l+1} = 11\prod_1^\infty (1 - x^{11m})^{-1}\{11AC^2 - 11^2C + 2AC - 32C - \tag{2.7}$$

From (2.7) we deduce, by comparison of coefficients in the two members, the correctness of Ramanujan's conjecture (1.1) for $q = 11$.

The identity (2.5) is equivalent to (2.7) and is far more convenient, since it involves only modular functions. We shall obtain all identities in the form (2.5), the transition to the partition function then being made by analogues of (2.6).

I. The Identities.

3. Preliminaries. By the statement "$F(\tau)$ is an entire modular function on a subgroup Γ of the full modular group" we mean that F satisfies the following conditions:

(A) $F(\tau)$ is an analytic function of τ which is regular for $\Im\tau > 0$.

(B) $F(\tau)$ has at most polar singularities, measured in the uniformizing variable, at the parabolic vertices of Γ.

(C) For every substitution

$$V\tau = \frac{a\tau + b}{c\tau + d} \tag{3.1}$$

belonging to Γ, we have

$$F(V\tau) = F(\tau). \tag{3.2}$$

The subgroup of greatest interest to us is $\Gamma_0(q)$, defined by $c \equiv 0 \pmod q$ in (3.1), where q is a prime > 3. Because of the primality of q, the fundamental region of $\Gamma_0(q)$ has only two parabolic vertices, which we may take to be $\tau' = 0$, $\tau = i\infty$.

The uniformizing variable for $\tau = i\infty$ is evidently $x = \exp 2\pi i\tau$, since $\Gamma_0(q)$ contains the substitution $S\tau = \tau + 1$. For $\tau' = 0$, we use the substitution $\tau' = -1/q\tau$ and employ the transformation equation of (2.2). This will give us an expansion in x, so that x is a uniformizing variable also for $\tau' = 0$.[8]

Notation. τ is a complex variable with $\Im\tau > 0$. The point $\tau = i\infty$ is the infinite point of the τ-sphere considered as the limit of iy, $y \to +\infty$, $y > 0$. $x = \exp 2\pi i\tau$. q is a prime > 3. S, T, V, W, $\cdots$, are linear substitutions, $\tau' = (a\tau + b)/(c\tau + d)$, with integral a, b, c, d and determinant $ad - bc = 1$. $\Gamma(1)$ is the full modular group, consisting of all such substitutions. $\Gamma_0(n)$ is the subgroup of $\Gamma(1)$ defined by $c \equiv 0 \pmod n$. $O(x^s)$ denotes a power series in x with *integral* coefficients beginning with a term of order $\geq s$.

Formulae.

$$\prod_1^\infty (1 - x^m) = \sum_{-\infty}^{+\infty} (-1)^n x^{n(3n+1)/2} = 1 - x - x^2 + x^5 + x^7 - - + + \cdots. \tag{3.3}$$

$$\prod_1^\infty (1 - x^m)^{-1} = \sum_{n=0}^\infty p(n) x^n, \quad p(0) = 1. \tag{3.4}$$

$$\eta(\tau + 1) = \exp(\pi i/12) \cdot \eta(\tau), \qquad \eta(\tau + 24) = \eta(\tau). \tag{3.5}$$

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\eta(\tau), \tag{3.6}$$

where the square-root is positive when τ is purely imaginary.

For every modular substitution $V\tau = (a\tau + b)/(c\tau + d) \in \Gamma(1)$, with $c > 0$, we have

$$\eta(V\tau) = \exp\left\{-\pi i\left(s(a, c) - \frac{a + d}{12c}\right)\right\}[-i(c\tau + d)]^{\frac{1}{2}}\eta(\tau), \tag{3.7}$$

[8] The substitutions of the group determine the form of the uniformizing variable. Thus, for $\Gamma_0(q)$, the uniformizing variable for $\tau = 0$ is $x' = \exp(-2\pi i/q\tau)$. It is only because the functions considered here have an additional property of the type (2.2) that $x = \exp 2\pi i\tau$ can be used in place of x', and it is convenient to do this.

where the sign of the square-root is determined as in (3.6), and where $s(a, c)$ is a "Dedekind sum." The Dedekind sums are fully discussed in [13].

(3.7) is the transformation formula of $\eta(\tau)$ and was discovered by Dedekind and Hermite. (3.6) is a special case of this. (3.5) is obvious from the definition (2.4) of $\eta(\tau)$, and the first two formulae are classical.

4. Modular functions of $\Gamma_0(q)$. Our first object will be to develop a set of modular invariants of $\Gamma_0(11)$. We shall, however, speak more generally of $\Gamma_0(q)$, q being always a prime > 3. We consider, first, modular *forms* of negative dimension, from which modular *functions* (zero dimension) will be obtained by division. Since we need *entire* modular functions, we must have one modular form which is zero-free in the interior of the fundamental region. Such a form is provided by

THEOREM 1. *The functions*

$$\psi_{q,r}(\tau) = \{\eta(q\tau)\eta(\tau)\}^r \tag{4.1}$$

where

$$r(q+1) \equiv 0 \pmod{24}, \tag{4.11}$$

are regular and zero-free in the upper half τ-plane. They have the transformation equations

$$\psi_{q,r}(V\tau) = (a/q)^r(c\tau+d)^r\psi_{q,r}(\tau), \quad V\tau \in \Gamma_0(q). \tag{4.12}$$

$$\psi_{q,r}(-1/q\tau) = (-iq^{1/2}\tau)^r\psi_{q,r}(\tau). \tag{4.13}$$

(a/q) is the Legendre-Jacobi symbol. It will be indicated in this way whenever confusion with the rational number a/q is not to be feared. Otherwise, we shall use the symbol $(a|q)$ or $\dfrac{a}{(q)}$.

Theorem 1 is proved in Part II.

A second set of modular forms, of dimension $-2r$, is given by the Eisenstein series [9, 10]

$$g_r(\tau) = \frac{(2r-1)!}{(2\pi)^{2r}} \sum_{m,n=-\infty}^{+\infty}{}' (m\tau+n)^{-2r}, \qquad r \geqq 2 \tag{4.21}$$

with the well-known property [11]

$$g_r(V\tau) = (c\tau+d)^{2r}g_r(\tau), \quad V\tau \in \Gamma(1). \tag{4.22}$$

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[9] Σ' means in this case that the term $m = n = 0$ is to be omitted.

[10] These functions can be defined for $r = 1$, also, even though the series on the right is no longer absolutely convergent. See [7], p. 207

[11] [3], pp. 20-22.

They have the Fourier series [12]

$$g_r(\tau) = B_r/2r + 2(-1)^r \sum_{n=1}^{\infty} \sigma_{2r-1}(n) x^n, \tag{4.23}$$

$$\sigma_l(n) = \sum_{d|n} d^l, \quad B_1 = 1/6, \quad B_2 = 1/30, \cdots.$$

The forms (4.21) belong to the full group $\Gamma(1)$ whereas we are interested in functions having the modular property on no group containing $\Gamma_0(q)$ as a proper subgroup. But consider

$$G_{q,r}(\tau) = q^r g_r(q\tau) - g_r(\tau), \qquad r \geq 2. \tag{4.31}$$

Clearly

$$G_{q,r}(V\tau) = (c\tau + d)^{2r} G_{q,r}(\tau), \quad V\tau \in \Gamma_0(q), \tag{4.32}$$

$$G_{q,r}(-1/q\tau) = -(q^{1/2}\tau)^{2r} G_{q,r}(\tau). \tag{4.33}$$

Still another class of modular forms can be obtained from the multiple theta-series. We assume from now on $q \equiv 3 \pmod 4$. The function

$$\theta_q(\tau) = \sum_{m,n=-\infty}^{+\infty} x^{Q(m,n)}, \quad Q = m^2 + mn + \frac{q+1}{4} n^2 \tag{4.41}$$

is a modular form of dimension -1 with the properties [12]

$$\theta(V\tau) = (c\tau + d)(a/q)\theta(\tau), \qquad (a/q) = \text{Legendre-Jacobi symbol}, \tag{4.42}$$

$$\theta(-1/q\tau) = -iq^{1/2}\tau\theta(\tau). \tag{4.43}$$

For higher dimensions we use either powers of θ_q or theta-series in more variables.[13]

From these modular forms we construct entire modular functions by dividing each form by a $\psi_{q,r}(\tau)$ of appropriate dimension. In particular, we obtain the functions of the following theorem.[14]

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[12] A convenient method for treating the transformation of θ_q is developed in Hecke [8]. We first reduce Q to a sum of squares: $Q = \frac{1}{4}((2m+n)^2 + qn^2) = \frac{1}{4}(M^2 + qN^2)$, where $M \equiv N \pmod 2$. Hence $Q = \mu\mu'/q$, where μ belongs to the principal ideal $(\sqrt{-q})$ in $K(\sqrt{-q})$. We then have $\theta_q(\tau) = \theta(\tau; 0, (1), \sqrt{-q})$ in Hecke's notation (p. 222). The relations (4.42), (4.43) of the text are equivalent to Sätze 7 and 5, respectively, of Hecke's paper. If $q \equiv 1 \pmod 4$, we must use quadratic forms of determinant $-4q$, rather than $-q$.

[13] The transformation of theta-series in more than two variables can be reduced to that of double theta-series. See [7], pp. 210-11.

[14] The functions which can be formed in this way from modular forms are not necessarily linearly independent. The number of linearly independent modular functions having at most a given valence can be determined from the Riemann-Roch Theorem applied to the Riemann surface which is the map of the fundamental region

THEOREM 2. *The functions*

$$A_q(\tau) = \frac{\theta_q{}^r(\tau)}{\psi_{q,r}(\tau)}, \quad B_q(\tau) = \frac{G_{q,r}(\tau)}{\psi_{q,2r}(\tau)} \tag{4.51}$$

are entire modular functions belonging to the subgroup $\Gamma_0(q)$:

$$A_q(V\tau) = A_q(\tau), \quad B_q(V\tau) = B_q(\tau), \quad V\tau \in \Gamma_0(q), \quad q \equiv 3(4). \tag{4.52}$$

Furthermore,[15]

$$A_q(-1/q\tau) = A_q(\tau), \qquad B_q(-1/q\tau) = -B_q(\tau). \tag{4.53}$$

The valences of A_q *and* B_q *are* $2s_q$ *and* $4s_q$, *respectively, where* $s_q = r(q+1)/24$.

We need to prove only the statements concerning valence. The valence of a modular function is defined to be the multiplicity of the root of the equation

$$F(\tau) = c,$$

where c is an arbitrary complex number and τ ranges over a point set consisting of the interior of the fundamental region and a complete set of inequivalent boundary points. The multiplicity must be measured in terms of the uniformizing variable. We may take $c = \infty$. Since $A_q(\tau)$ and $B_q(\tau)$ are entire modular functions, they can have poles only at the parabolic points $\tau' = 0$, $\tau = i\infty$.

We can find the expansions of A_q and B_q from (4.1), (4.11), (2.4), (3.3), (3.4), (4.31), (4.23), and

$$\theta_q(\tau) = \sum_{n=0}^{\infty} A_Q(n) x^n, \qquad x = \exp 2\pi i\tau, \tag{4.61}$$

where $A_Q(n)$ is the number of representations[16] of n by the form Q. Thus, for $\tau = i\infty$,

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of $\Gamma_0(q)$. Since $\Gamma_0(11)$ is of genus one, it follows from the Riemann-Roch Theorem that there are $m - 1 + 1 = m$ independent functions on $\Gamma_0(11)$ of valence at most m if $m > 1$. In particular, there are 2 independent 2-valent functions (since univalent functions do not exist on Riemann surfaces of positive genus 1) which may be taken as $A_{11}(\tau)$ and $A_{11}(\tau) + 1$. Now, we can construct another 2-valent function, namely, $G_{11,1}(\tau)/\psi_{11,2}(\tau) = A'_{11}$, where $G_{11,1} = 11g_1(11\tau) - g_1(\tau)$ (see footnote 10). Hence there must be a linear relation between A_{11} and A'_{11}. Indeed, we find by comparing power series that $6A'_{11} = 5A_{11} - 8$. Similar relations hold among the m-valent functions when $m > 2$.

[15] Expressed otherwise, $A_q(\tau)$ is an automorphic function of the group $\Gamma(q) = \Gamma_0(q) + W \cdot \Gamma_0(q)$, $W = -1/q\tau$. Of course, $\Gamma(q)$ is not a subgroup of the modular group.

[16] Counted in the usual way, i. e., two representations are distinct if they differ only in the order or in the sign of the unsquared summands. For special $A_Q(l)$ explicit formulae are known: for $Q = m^2 + mn + 3n^2$, $\det Q = -11$, we have

$$(4.62)\quad A_q(\tau) = \exp\{-(\pi i\tau/12)r(q+1)\} + \cdots = x^{-s_q} + \cdots,$$

$$(4.63)\quad B_q(\tau) = \exp\{-(\pi i\tau/12)\cdot 2r(q+1)\} + \cdots = x^{-2s_q} + \cdots.$$

At the other parabolic point, we use $\tau' = -1/q\tau$ and (4.53), and have

$$(4.64)\quad A_q(\tau') = A_q(\tau) = x^{-s_q} + \cdots,\quad B_q(\tau') = -B_q(\tau) = -x^{-2s_q} + \cdots.$$

Hence the valences are $2s_q$ for A_q and $4s_q$ for B_q, as stated.

We conclude this section with a proof of the fact, noted in **2**, that $U_q\Phi(\tau; q^2)$ is an entire modular function of $\Gamma_0(q)$. We recall the definition

$$(4.71)\qquad U_q\Phi(\tau; q^2) = q^{-1}\sum_{\lambda=0}^{q-1}\Phi\left(\frac{\tau+\lambda}{q};\ q^2\right).$$

Condition (A) in the definition of an entire modular function — **3** — is certainly fulfilled since $U_q\Phi$ is simply a sum of functions, each of which is regular in the upper half-plane.

In the sum of (4.71) we can replace the range of summation of λ by the equivalent condition: λ modulo q. This follows from the fact that Φ is periodic with period 1; replacing λ by $\lambda + q$ leaves each summand unaltered. Hence

$$(4.72)\qquad U_q\Phi(\tau; q^2) = q^{-1}\sum_{\lambda}\Phi\left(\frac{\tau+24\lambda}{q};\ q^2\right),\qquad \lambda \bmod q.$$

Using (2.4), (3.3), (3.4), we find

$$(4.73)\qquad \Phi(\tau; q^2) = \frac{\eta(q^2\tau)}{\eta(\tau)} = x^u(1 + x + 2x^2 + \cdots)$$

$$= \sum_{n=u}^{\infty}\phi_n x^n,\quad u = \frac{q^2-1}{24},\qquad \phi_u \neq 0.$$

Introducing (4.73) in (4.72), we have

$$U_q\Phi(\tau; q^2) = q^{-1}\sum_{\lambda,n}\phi_n \exp\left(2\pi i n(\tau + 24\lambda/q)\right),\quad \lambda \bmod q,\quad n \geq u$$

$$= q^{-1}\sum_{n}\phi_n \exp\left(2\pi i(n\tau/q)\right)\sum_{\lambda}\exp\left(2\pi i n(24\lambda/q)\right)$$

$$= \sum_{m}\phi_m \exp 2\pi i(m\tau/q),\quad q\,|\,m.$$

$$(4.74)\quad U_q\Phi(\tau; q^2) = \sum_{n=v}^{\infty}\phi_{qn}\exp(2\pi i n\tau) = \sum_{n=v}^{\infty}\phi_{qn}x^n,\quad v = [u/q] + 1.$$

$A_Q(l) = 2\{\sum_{d_1|l} d_1 - \sum_{d_2|l} d_2\}$, where $d_1 \equiv$ quad. residue, $d_2 \equiv$ quad. non-residue, (mod 11), and $l \not\equiv 0(11)$. See Dickson, *Introduction to the Theory of Numbers*, 1929, p. 81, ex. 12. For $l = 11k$, we use the substitution $m = M + 6N$, $n = -2M - N$, and deduce easily $A_Q(11k) = A_Q(k)$. Hence for $l = 11^\beta k_1$, $11 \nmid k_1$, $A_Q(l) = A_Q(k_1)$.

Thus for $\tau = i\infty$, $U_q\Phi$ has an expansion of the required type. The effect of U_q is simply to replace a_n by a_{qn}.

For $\tau' = 0$, we write $\tau' = T = -1/\tau$ and apply the transformation equations (3.6) and (3.7). First,

$$U_q\Phi(\tau; q^2) = q^{-1}\sum_{\lambda}\eta(q(\tau+24\lambda))\eta^{-1}(\tau+24\lambda/q)$$
$$= q^{-1}\eta(q\tau)\sum_{\lambda}\eta^{-1}(T_{24\lambda}\tau), \qquad \lambda \bmod q$$

where $T_\lambda\tau = (\tau+\lambda)/q$ and where we have made use of (3.5). Hence,

$$qU_q\Phi(\tau'; q^2) = \eta(-q/\tau)\sum_{\lambda=0}^{q-1}\eta^{-1}(T_{24\lambda}T\tau) = \eta(-q/\tau)\{\eta^{-1}(-1/q\tau) + \sum_{\lambda=1}^{q-1}\eta^{-1}(T_{24\lambda}T\tau)\}. \tag{4.81}$$

Now

$$T_{24\lambda}T = V_\mu T_{24\mu}, \tag{4.82}$$

where

$$V_\mu = \begin{pmatrix} 24\lambda & -(24^2\lambda\mu+1)/q \\ q & -24\mu \end{pmatrix} \tag{4.83}$$

$$24^2\lambda\mu \equiv -1 \pmod q, \qquad 0 < \mu < q. \tag{4.84}$$

We see from (4.84) that μ runs with λ over a reduced residue system modulo q. Using (4.82) in (4.81) and applying the transformation equation of $\eta(\tau)$, we get

$$qU_q\Phi(\tau'; q^2) = q^{-1}\eta(\tau/q)\eta^{-1}(q\tau) + q^{-1/2}\eta(\tau/q)\sum_{\mu=1}^{q-1}\eta^{-1}(V_\mu T_{24\mu}\tau)$$
$$= q^{-1}\Phi^{-1}(\tau/q; q^2) + q^{-1/2}\eta(\tau/q)\sum_{\mu=1}^{q-1}C_\mu\eta^{-1}(T_{24\mu}\tau), \tag{4.85}$$

with

$$C_\mu = \exp\{\pi i(s(24\lambda, q) - 2(\lambda-\mu)/q)\}.$$

A computation similar to [12], (5.4), shows that

$$C_\mu = i^{(1-q)/2}(6\mu/q) = \epsilon_1(\mu/q), \qquad \epsilon_1{}^4 = 1. \tag{4.86}$$

Hence, the sum in the last member of (4.85) is

$$\sum_\mu = \epsilon_1\sum_{n,\mu}\exp\left(-\frac{\pi i}{12}\,\frac{\tau+24\mu}{q}\right)\cdot(\mu/q)p(n)\exp\left(2\pi i n\,\frac{\tau+24\mu}{q}\right), \quad n \geq 0, \quad (\mu = 1, \cdots, q-1)$$
$$= \epsilon_1\exp(-\pi i\tau/12q)\sum_{n=0}^{\infty}p(n)\exp(2\pi i(n\tau/q))\sum_{\mu=0}^{q-1}(\mu/q)$$
$$\times\exp\{2\pi i\cdot(24n-1)\mu/q\}\text{—since } (0/q) = 0\text{—}$$

$$= \epsilon_2 q^{1/2} \exp(-\pi i\tau/12q) \sum_{n=0}^{\infty} \left(\frac{24n-1}{q}\right) p(n) \exp(2\pi i(n\tau/q), \qquad \epsilon_2{}^4 = 1$$

where we have made use of the Gaussian sum

$$(4.87) \qquad G(h,q) = \sum_l (l/q) \exp(2\pi i(hl/q) = \epsilon_3 q^{1/2}(h/q), \qquad l \bmod q, \quad \epsilon_3{}^4 = 1.$$

We replace τ by $q\tau$ and obtain finally

$$q^2 U_q \Phi(\tau'; q^2) = \Phi^{-1}(\tau; q^2) + \epsilon_2 \eta(\tau) \exp(-\pi i\tau/12)(1 + O(x))$$
$$= O(x^{-u}), \qquad \tau' = -1/q\tau,$$

by (4.73) and (2.4). Hence, $U_q\Phi$ has a pole of order u in $x = \exp 2\pi i\tau$ for $\tau' = 0$, and condition (B) is satisfied.

That $U_q\Phi$ fulfills (C) with $\Gamma = \Gamma_0(q)$ is seen as follows. From Theorem 3, which is proved in Part II, we take the fact ($\alpha = 2$) that

$$(4.91) \qquad \Phi(V\tau; q^2) = \Phi(\tau; q^2), \qquad V \in \Gamma_0(q^2).$$

We require also the following lemma, proved in Part II.

LEMMA 1. *To any $V \in \Gamma_0(q)$ and λ, $0 \leq \lambda < q$, there exist μ, $0 \leq \mu < q$, and $W_\mu \in \Gamma_0(q^2)$, such that*

$$(4.92) \qquad T_\lambda V = W_\mu T_\mu, \qquad T_\lambda \tau = (\tau + \lambda)/q.$$

Furthermore, μ runs with λ over a complete residue system modulo q.

Therefore,

$$(4.93) \qquad U_q\Phi(V\tau; q^2) = q^{-1} \sum_\lambda \Phi((V\tau + \lambda)/q; q^2), \qquad \lambda \bmod q$$
$$= q^{-1} \sum_\lambda \Phi(T_\lambda V\tau; q^2) = q^{-1} \sum_\mu \Phi(W_\mu T_\mu \tau; q^2), \qquad \mu \bmod q$$
$$= q^{-1} \sum_\mu \Phi(T_\mu \tau; q^2) = U_q\Phi(\tau; q^2),$$

as required. This proves (C) and

THEOREM 4. *The function $U_q\Phi(\tau; q^2)$ is an entire modular function of the group $\Gamma_0(q)$.*

5. A rational basis for $\Gamma_0(q)$. The next step is to find a basis for the entire modular functions of $\Gamma_0(q)$. For this purpose the following lemma will be useful.[17]

LEMMA 2. *Let $f_1(\tau)$, $f_2(\tau)$ be modular functions on a group Γ having*

[17] For proof see [1], Th. 13, pp. 96-97.

the property that an arbitrary pair of values of the functions is assumed at but one point of each fundamental region of Γ, i. e., *the equations*

$$f_1(\tau) = f_1(\tau'), \qquad f_2(\tau) = f_2(\tau')$$

shall imply that τ *and* τ' *are congruent under* Γ. *Then every modular function of* Γ *is a rational function of* f_1 *and* f_2.

We are now ready to prove

THEOREM 5. *If the function* $A_q(\tau)$ *of* (4. 51) *is 2-valent on* $\Gamma_0(q)$, *then the functions* $A_q(\tau)$, $B_q(\tau)$ *constitute a rational basis for* $\Gamma_0(q)$.

Proof. By Lemma 2 we have only to show that

$$A_q(\tau) = A_q(\tau'), \qquad B_q(\tau) = B_q(\tau') \tag{5. 1}$$

is impossible if $\tau \not\equiv \tau'$ and τ, τ' lie in the same fundamental region of $\Gamma_0(q)$. Since $A_q(\tau)$ is 2-valent it follows from (5. 1), (4. 53) that $\tau' = -1/q\tau$. But then

$$B_q(\tau) = B_q(-1/q\tau),$$

which, in general, is inconsistent with (4. 53), since $B_q(\tau)$ is not identically zero.[18]

As a special case we take $q = 11 \equiv 3 \pmod 4$, $n = 2$, $s_{11} = 1$, satisfying the conditions of Theorem 2. Then $A_{11}(\tau)$ is 2-valent on $\Gamma_0(11)$ and we have

COROLLARY.[19] *The functions* $A_{11}(\tau)$, $B_{11}(\tau)$ *form a rational basis for the entire modular functions of* $\Gamma_0(11)$.

Note that

$$\text{val } A_{11} = 2, \qquad \text{val } B_{11} = 4. \tag{5. 2}$$

6. An identity for modulus eleven. Since $U_{11}\Phi(\tau; 11^2)$ is an entire modular function of the group $\Gamma_0(11)$ (Theorem 4), it is a rational function of $A_{11}(\tau)$, $B_{11}(\tau)$:

$$U_{11}\Phi = P(A_{11}, B_{11}). \tag{6. 1}$$

It will turn out, however, that P is a polynomial. That is, we assume P as a polynomial with undetermined coefficients, and then show how to determine these coefficients so that the difference $U_{11}\Phi - P$, which is an entire modular

[18] This argument shows, incidentally, that $B_q(\tau)$ must vanish at the fixed point of the substitution $\tau' = -1/q\tau$, and indeed at the fixed points of all elliptic substitutions of the group $\Gamma(q)$ (see footnote 15).

[19] See [2], p. 406, (12). Fricke denotes $A(\tau)$ by $\tau(\omega)$ and $B(\tau)$ by $\sigma(\omega)$.

function, has no poles. Then it must be a constant, which can be evaluated at $\tau = i\infty$, say.

The expansions of $U_{11}\Phi$ at the parabolic points were worked out in the proof of Theorem 4. We repeat them here.

$$U_{11}\Phi(\tau; 11^2) = O(x) \tag{6.2}$$

$$11^2 U_{11}\Phi(\tau'; 11^2) = x^{-5} - x^{-4} - x^{-3} + O(1), \qquad \tau' = -1/11\tau, \tag{6.3}$$

where we have used the expansion

$$\Phi^{-1}(\tau; 11^2) = \frac{\eta(\tau)}{\eta(121\tau)} = x^{-5}(1 - x - x^2 + x^5 + \cdots) \cdot \sum_{n=0}^{\infty} p(n)x^{121n},$$

calculated from (2.4), (3.3), (3.4).

We are now in a position to compare the principal parts of the two members of (6.1). At $\tau = i\infty$, $U_{11}\Phi$ has a zero of the first order, whereas $A_{11} = A$ and $B_{11} = B$ both have poles, by (4.62), (4.63). It will therefore be convenient to construct a new function, $C_{11}(\tau) = C_{11}$, which has a zero of the first order at $\tau = i\infty$. We wish C to have the lowest possible valence, i.e., 2, since the fundamental region of $\Gamma_0(11)$, considered as a closed Riemann surface, is of genus one. In other words, C must have a double pole at $\tau' = 0$. If we add the requirement that the expansion of C at $\tau = i\infty$ shall begin with x, the function is then determined, by the Riemann-Roch Theorem.[20]

From the expansions—see the table on p. 514—

$$A(\tau) = x^{-1} + 6 + 17x + 46x^2 + \cdots \tag{6.41}$$

$$B(\tau) = x^2 + 2x^{-1} - 12 - 116x - 597x^2 + \cdots, \tag{6.42}$$

we find

$$C(\tau) = 2^{-1} \cdot 11^{-2}(A^2(\tau) - 10A(\tau) - 22 - B(\tau)). \tag{6.43}$$

Its expansion at $\tau = i\infty$ is

$$C(\tau) = x + 5x^2 + \cdots = \sum_{n=1}^{\infty} c_n x^n. \tag{6.44}$$

From the last relation and (4.53), we see that

$$\begin{aligned} 2 \cdot 11^2 C(\tau') &= A^2(\tau') - 10A(\tau') - 22 - B(\tau'), \quad \tau' = -1/11\tau \\ &= A^2(\tau) - 10A(\tau) - 22 + B(\tau). \end{aligned} \tag{6.45}$$

---

[20] The number of linearly independent functions having a pole of second order at $\tau = 0$ is $2 - 1 + 1 = 2$. The most general such function contains two arbitrary constants, one of which is used to insure the zero at $\tau = i\infty$, the other to guarantee the coefficient 1 for the linear term.

This gives [21]

$$11^2C(\tau') = x^{-2} + 2x^{-1} + \cdots = \sum_{n=-2}^{\infty} c'_n x^n. \tag{6.46}$$

LEMMA 3. *The coefficients c_n, c'_n of* (6.44), (6.46) *are rational integers.*

Proof. Since A, B both belong to $\Gamma_0(11)$, they are connected by an algebraic relation, which is of degree 2 in B and of degree 4 in A, in accordance with the valences (5.2). By writing such a relation with undetermined coefficients and then using the expansions (6.41), (6.42) we find easily

$$B^2 = A^4 - 20A^3 + 56A^2 - 44A. \tag{6.51}$$

From [21] (6.43)

$$\begin{aligned}(2\cdot 11^2C + B)^2 &= (A^2 - 10A - 22)^2 = A^4 - 20A^3 + 56A^2 - 44A + 484(A\\ &\equiv B^2 \pmod{2^2\cdot 11^2},\\ 2\cdot 11^2C + B &\equiv B \pmod{2\cdot 11^2}\\ \text{or}\quad 2\cdot 11^2C &\equiv 0 \pmod{2\cdot 11^2}.\end{aligned}$$

Thus the c_n are integers.

Again, let $C(-1/11\tau) = C'$, etc. Then, from (6.45),

$$\begin{aligned}2\cdot 11^2C' - B &= A^2 - 10A - 22,\\ (2\cdot 11^2C' - B)^2 &= (A^2 - 10A - 22)^2 \equiv B^2 \pmod 4,\\ 2\cdot 11^2C' - B &\equiv B \pmod 2,\ 11^2C' \equiv B \pmod 1.\end{aligned}$$

This proves the lemma.[22]

For future reference we include another lemma.

LEMMA 4. $AC - 1 \equiv 0 \pmod{11}$.

Proof. First,

$$\begin{aligned}22A^5 - 55A^4 + 44A^3 &\equiv 0 \pmod{11}\\ \text{or}\quad A^4(A^2 + 2A + 1) &\equiv A^2(A^4 - 20A^3 + 56A^2 - 44A) \pmod{11}\\ &\equiv A^2B^2 \pmod{11}.\end{aligned}$$

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[21] In what follows we employ the following simple theorem: Let $F = x^{-l} + \Sigma a_\nu x^\nu$, $G = x^{-l} + \Sigma b_\nu x^\nu$, $\nu > -l$; a_ν *and* b_ν *are integers. Then if* $F^2 \equiv G^2 \pmod k$, k *odd, we have* $F \equiv G \pmod k$; *if* $F^2 \equiv G^2 \pmod{2^\lambda}$, $\lambda > 0$, *then* $F \equiv G \pmod{2^{\lambda-1}}$. In general, we would have $F \equiv \pm G$. The correct sign can be determined from the first terms of the power series for F and G.

[22] The functions C and $C' = C(-1/11\tau)$ do not satisfy so simple a transformation equation as A and B do. However, this equation can be found easily by eliminating A and B from the three relations (6.43), (6.45), and (6.51). The result is

$$11^3C^2C'^2 - 12.11CC' - 11(C + C') - 1 = 0.$$

Hence,

$$A^2(A+1) \equiv AB \pmod{11}$$
$$2^2 \cdot 11^2 A^2(A+1) \equiv 2^2 \cdot 11^2 AB \pmod{2^2 \cdot 11^3}$$
$$A^2(B^2 + 2^2 \cdot 11^2(A+1)) \equiv 2^2 \cdot 11^4 + 2^2 \cdot 11^2 AB + A^2B^2 \pmod{2^2 \cdot 11^3}$$
$$A^2(A^2 - 10A - 22)^2 \equiv (2 \cdot 11^2 + AB)^2 \pmod{2^2 \cdot 11^3}$$

if we use the computation in the lines following (6.51). Hence, taking note of footnote [21], we obtain

$$A(A^2 - 10A - 22) \equiv 2 \cdot 11^2 + AB \pmod{2 \cdot 11^3}$$
$$A(A^2 - 10A - 22 - B) \equiv 2 \cdot 11^2 \pmod{2 \cdot 11^3},$$

and by (6.43)

$$2 \cdot 11^2 AC \equiv 2 \cdot 11^2 \pmod{2 \cdot 11^3},$$

which completes the proof.

Following the procedure outlined at the beginning of this section, we set $U_{11}\Phi$ equal to a polynomial P in A and C with undetermined coefficients. The poles of even order in the expansion (6.3) of $U_{11}\Phi$ at $\tau' = 0$ can be eliminated by powers of C, those of odd order by A times a power of C. Since both $U_{11}\Phi$ and P are bounded at $\tau = i\infty$, it is clear that the term in x^{-1} (at $\tau' = 0$) must drop out automatically. Otherwise, the difference $U_{11}\Phi - P$ would be an entire modular function having a single pole of the first order on $\Gamma_0(11)$, a subgroup of genus one.

We therefore write

$$11^2 U_{11}\Phi(-1/11\tau; 11^2) = x^{-5} - x^{-4} - x^{-3} + O(1)$$
$$= a_{12}AC'^2 + a_{02}C'^2 + a_{11}AC' + a_{01}C' + a_{00}, \qquad C' = C(-1/11\tau)$$
$$11^2 U_{11}\Phi(\tau; 11^2) = O(x) = a_{12}AC^2 + a_{02}C^2 + a_{11}AC + a_{01}C + a_{00}.$$

This gives us the following five equations, which are solved seriatim beginning with the uppermost.

$$\begin{array}{rrrrr} a_{12} \cdot 11^{-4} & & & & = 1 \\ a_{12} \cdot 10 \cdot 11^{-4} + a_{02} \cdot 11^{-4} & & & & = -1 \\ a_{12} \cdot 21 \cdot 11^{-4} + a_{02} \cdot 4 \cdot 11^{-4} & + a_{11} \cdot 11^{-2} & & & = -1 \\ -a_{12} \cdot 4 \cdot 11^{-3} & -a_{02} \cdot 20 \cdot 11^{-4} + a_{11} \cdot 8 \cdot 11^{-2} & + a_{01} \cdot 11^{-2} & & = 0 \\ & a_{11} & & + a_{00} & = 0 \end{array} \tag{6.52}$$

The values of the a's are: $a_{12} = 11^2$, $a_{02} = -11^3$, $a_{11} = 2 \cdot 11$, $a_{01} = -32 \cdot 11$, $a_{00} = -2 \cdot 11$.

This constitutes the proof of

THEOREM 6. *For $q = 11$, there is a "Ramanujan identity"*

$$U_{11}\Phi(\tau; 121) = 11(11AC^2 - 11^2C + 2AC - 32C - 2). \tag{6.6}$$

Ramanujan's congruence (1.3) *with* $q^a = 11$ *follows from* (6.6) *and* (2.6), *which together yield* (2.7).

7. An induction. We now wish to establish identities of the same sort as (6.6) but involving higher powers of 11. But it is not yet clear what functions, analogous to $U_{11}\Phi(\tau; 121)$, should appear in the left member of such an identity. Our guide in this respect will be formula (2.6), which connects the partition function with $U_{11}\Phi$. The first step, therefore, will be an appropriate generalization of this formula.

THEOREM 7. *Let*

$$L(\tau; 1) = 1, \tag{7.11}$$

$$L(\tau; q^{2\beta+1}) = U_q\{\Phi(\tau; q^2) \cdot L(\tau; q^{2\beta})\}, \qquad \beta \geq 0 \tag{7.12}$$

$$L(\tau; q^{2\beta+2}) = U_q L(\tau; q^{2\beta+1}), \qquad \beta \geq 0. \tag{7.13}$$

Then for $\alpha \geq 1$,

$$L(\tau; q^\alpha) = x^t \prod_1^\infty (1 - x^{\epsilon m}) \sum_{l=0}^\infty p(q^\alpha l + \rho) x^l, \qquad x = \exp 2\pi i\tau, \tag{7.2}$$

where [23]

$$t = t(q^\alpha) = 1 \text{ for } q < 24, \tag{7.21}$$

$$\epsilon = \begin{cases} 1, & \alpha \text{ even} \\ q, & \alpha \text{ odd} \end{cases} \tag{7.22}$$

and

$$24\rho \equiv 1 \pmod{q^\alpha}, \qquad 0 < \rho < q^\alpha. \tag{7.23}$$

(2.6) is the case $\alpha = 1$ of (7.2).

Proof. We replace (7.2) by two statements,

$$L(\tau; q^\alpha) = q^{-\alpha} \sum_\lambda \eta(\epsilon\tau)\eta^{-1}\left(\frac{\tau + 24\lambda}{q^\alpha}\right), \qquad \lambda \bmod q^\alpha, \tag{7.31}$$

$$x^t \prod_1^\infty (1 - x^{\epsilon m}) \sum_{l=0}^\infty p(q^\alpha l + \rho) x^l = q^{-\alpha} \sum_\lambda \eta(\epsilon\tau)\eta^{-1}\left(\frac{\tau + 24\lambda}{q^\alpha}\right), \tag{7.32}$$

and proceed to prove the first. Suppose $\alpha = 1$; then $\beta = 0$, and (7.31) reduces to

$$U_q\Phi(\tau; q^2) = q^{-1} \sum_\lambda \eta(q\tau)\eta^{-1}\left(\frac{\tau + 24\lambda}{q}\right), \qquad \lambda \bmod q. \tag{7.4}$$

[23] Actually, the value of t is 1 if α is even and $[q/24] + 1$ if α is odd.

Now (see the lines following (4.71))

$$U_q\Phi(\tau;q^2) = q^{-1}\sum_\lambda \Phi\left(\frac{\tau+24\lambda}{q}\,;\,q^2\right), \qquad \lambda \bmod q,$$

$$= q^{-1}\sum_\lambda \eta(q(\tau+24\lambda))\eta^{-1}\left(\frac{\tau+24\lambda}{q}\right) = q^{-1}\sum_\lambda \eta(q\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q}\right),$$

where we have used (3.5). This is (7.4).

Suppose now that (7.31) has been proved for $\alpha = 2\beta$. From (7.12) we have

$$L(\tau;q^{2\beta+1}) = U_q\left\{\frac{\eta(q^2\tau)}{\eta(\tau)}\cdot q^{-2\beta}\sum_\lambda \eta(\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q^{2\beta}}\right)\right\}, \qquad \lambda \bmod q^a$$

$$= q^{-2\beta}U_q\left\{\eta(q^2\tau)\sum_\lambda \eta^{-1}\left(\frac{\tau+24\lambda}{q^{2\beta}}\right)\right\}$$

$$= q^{-2\beta-1}\sum_\mu \eta\left(q^2\frac{\tau+24\mu}{q}\right)\sum_\lambda \eta^{-1}\left(\left(\frac{\tau+24\mu}{q}+24\lambda\right)/q^{2\beta}\right)$$

$$= q^{-2\beta-1}\eta(q\tau)\sum_\nu \eta^{-1}\left(\frac{\tau+24\nu}{q^{2\beta+1}}\right), \qquad \nu \bmod q^{2\beta+1},$$

which establishes the inductive step from 2β to $2\beta+1$. The step from $2\beta+1$ to $2\beta+2$ is accomplished in the same way. Thus (7.31) is proved.

To demonstrate (7.32) we employ the formula

$$\frac{1}{\eta(\tau)} = \exp\left(-\frac{\pi i\tau}{12}\right)\cdot\sum_{n=0}^{\infty}p(n)x^n \tag{7.5}$$

which follows immediately from (2.4) and (3.4). Introducing this in the right member of (7.32) we see that it becomes

$$q^{-a}\eta(\epsilon\tau)\sum_\lambda \exp\left(-\frac{\pi i}{12}\cdot\frac{\tau+24\lambda}{q^a}\right)\sum_{n=0}^{\infty}p(n)\exp\left(2\pi in\,\frac{\tau+24\lambda}{q^a}\right), \qquad \lambda \bmod q^a$$

$$= q^{-a}\eta(\epsilon\tau)\exp\left(-\frac{\pi i\tau}{12q^a}\right)\sum_n p(n)\exp\left(2\pi i\frac{n\tau}{q^a}\right)\sum_\lambda \exp\left(2\pi i\frac{\lambda}{q^a}(24n-1)\right)$$

$$= \exp\left(-\frac{\pi i\tau}{12q^a}+\frac{\pi i\epsilon\tau}{12}\right)\prod_1^\infty(1-x^{\epsilon m})\sum_m p(m)\exp\left(2\pi i\,\frac{m\tau}{q^a}\right),$$

$$24m \equiv 1 \pmod{q^a},$$

from which (7.32) follows readily with

$$t = \frac{24\rho-1+\epsilon q^a}{24q^a}. \tag{7.6}$$

Clearly t is an integer. For, $\epsilon q^a = q^{2\beta} = (q^\beta)^2 \equiv 1 \pmod{24}$ for $\alpha = 2\beta$, and $\epsilon q^a = q^{a+1} = q^{2\beta+2} = (q^{\beta+1})^2 \equiv 1 \pmod{24}$ for $\alpha = 2\beta+1$; moreover,

q^a divides $24\rho - 1$ by (7.23). Since $0 < \rho < q^a$ and since we assumed in (7.21) that $q < 24$, we find that

$$0 < t < 1 + q/24 < 2;$$

therefore, as an integer, $t = 1$. This completes the proof of the theorem.

Theorem 7 shows us that we must attempt to construct identities in which one member is $L(\tau; 11^a)$ while the other is a polynomial in A and C. Obviously we shall proceed by induction, in view of the recursive definition (7.12), (7.13) of the L's. The first step of the induction ($\alpha = 1$, $\beta = 0$) has already been proved; it is Theorem 6.

The nature of the inductive step from α to $\alpha + 1$ will depend upon the parity of α, as the equations (7.12), (7.13) clearly show.[24] However, we can bring these two cases together. Suppose that we have an identity

$$L(\tau; 11^a) = P(A(\tau), C(\tau)), \tag{7.7}$$

where P is a polynomial in A and C with rational coefficients, in which A appears at most linearly and always multiplied by a power of C. In order to obtain $L(\tau; 11^{a+1})$, either we apply U_{11} directly to P or we first multiply by $\Phi(\tau; 11^2)$ and then operate with U_{11}. Since U_{11} is linear we may apply it to each term of P or ΦP separately. Thus the most general function which we need to consider is

$$G(\tau; i, j, k) = U_{11}\Phi^i(\tau; q^2)A^j(\tau)C^k(\tau), \qquad i = 0 \text{ or } 1, \quad j = 0 \text{ or } 1, \quad k \geq j. \tag{7.8}$$

The induction will be completely established if we prove that $G(\tau; i, j, k)$ can be expressed as a polynomial of type $P(A(\tau), C(\tau))$ in (7.7).

The procedure will parallel that used in the case $\alpha = 1$. The whole proof consists in showing that G is an entire modular function of $\Gamma_0(11)$. The expansion of G at $\tau = i\infty$ will not contain any terms with negative exponents. By examining the proof of Theorem 6 we see readily that any such function can be expressed in the form of a polynomial of the desired type.

8. Identities for powers of eleven.

THEOREM 8. *Let $G(\tau; i, j, k)$ be defined by (7.8). Then $G(\tau; i, j, k)$ is an entire modular function of $\Gamma_0(11)$. Moreover, we have*

$$G(\tau; i, j, k) = P(A, C)$$

where P is of the same type as in (7.7).

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[24] See [12], pp. 624-625.

Proof. We must show that conditions (A)-(C) of **3** are fulfilled. This is obvious for (A) since all the functions involved in G are regular in the upper τ-half-plane and U_{11} is simply a sum of such functions.

We now consider the expansions of G at the parabolic vertices. At $\tau = i\infty$, we have

$$(8.1) \qquad \Phi^i(\tau; 11^2) = O(x^{5i}), \quad A^j(\tau) = O(x^{-j}), \quad C^k(\tau) = O(x^k),$$

by (4.73), (6.41), (6.44). Then

$$\Phi^i A^j C^k(\tau) = O(x^w) = \sum_{n=w}^{\infty} d_n x^n, \qquad w \geq 0.$$

Hence, as we saw in (4.74),

$$(8.2) \qquad G(\tau; i, j, k) = U_{11}\Phi^i A^j C^k = \sum_{n=w'}^{\infty} d_{11n} x^n, \qquad w' \geq 1.$$

For $\tau' = 0$ we write $\tau' = T\tau = -1/\tau$, and have

$$\begin{aligned} G(\tau'; i, j, k) &= 11^{-1} \sum_{\lambda=0}^{10} \Phi^i A^j C^k(T_\lambda T\tau) \\ &= 11^{-1} \sum_{\lambda=0}^{10} \Phi^i A^j C^k(T_{24\lambda} T\tau), \qquad T_{\lambda\tau} = \frac{\tau + \lambda}{q} \\ &= 11^{-1}\Phi^i A^j C^k(-1/11\tau) + 11^{-1} \sum_{\lambda=1}^{10} \Phi^i A^j C^k(T_{24\lambda} T\tau) \\ &= 11^{-1}\Phi^i A^j C^k(-1/11\tau) + 11^{-1} \sum_{\mu=1}^{10} \Phi^i A^j C^k(V_\mu T_{24\mu}\tau), \end{aligned}$$

if we recall (4.82). The functions A and C are invariant under V_μ, which belongs to $\Gamma_0(11)$, but $\Phi(T_{24\mu}\tau)$ picks up the factor

$11^{-1/2}C_\mu = 11^{-1/2} i\,(\mu/11)$; where $(\mu/11)$ is the Legendre-Jacobi symbol,

as is seen if we put $q = 11$ in (4.81)-(4.86). This gives

$$(8.31) \quad 11G(\tau') = \Phi^i A^j C^k(-1/11\tau) + 11^{-1/2} i \sum_{\mu=1}^{10} (\mu/11)\Phi^i A^j C^k(T_{24\mu}\tau).$$

We write

$$(8.32) \quad \Phi^i(\tau; 11^2) = \sum_{n=5i}^{\infty} \phi_n x^n; \quad A^j(\tau) = \sum_{n=-j}^{\infty} a_n x^n, \quad C^k(\tau) = \sum_{n=k}^{\infty} c_n x^n,$$

where the coefficients are integers. Then

$$\begin{aligned} 11G(\tau') &= \Phi^i A^j C^k(-1/11\tau) \\ &\quad + 11^{-1/2} i \sum_{\mu,m,n,p} (\mu/11)\phi_m a_n c_p \exp\left\{ 2\pi i(m + n + p)\left(\frac{\tau + 24\mu}{11}\right)\right\}. \end{aligned}$$

The sum in the right member is

$$\begin{aligned} &\sum_{m,n,p} \phi_m a_n c_p \exp\left(2\pi i(m + n + p)(\tau/11)\right) \sum_{\mu=0}^{10} (\mu/11) \exp\{2\pi i(m + n + p)(24\mu/11)\} \\ &\qquad = \sum_{m,n,p} \phi_m a_n c_p \cdot \left(\frac{(m + n + p)24}{11}\right) 11^{\frac{1}{2}} \exp\{2\pi i(m + n + p)(\tau/11)\}, \end{aligned}$$

where we have evaluated a Gaussian sum, (4.87). Using (8.32) and replacing τ by 11τ, we now have $\tau' = -1/11\tau$,

$$11^{i+2k+1}G(\tau') = O(x^{-5i-11j-22k}) + O(x^{5i-j+k}) = O(x^{-5i-11j-22k}). \tag{8.4}$$

(8.2) and (8.4) show that G has only polar singularities in $x = \exp 2\pi i\tau$ at the parabolic vertices.

It remains to prove that

$$G(V\tau) = G(\tau), \qquad V \in \Gamma_0(11). \tag{8.5}$$

Here we require (4.92) of Lemma 1.

$$\begin{aligned} G(V\tau) &= 11^{-1}\sum_{\lambda=0}^{10}\Phi^i A^j C^k(T_\lambda V\tau), \qquad V \in \Gamma_0(11) \\ &= 11^{-1}\sum_{\mu=0}^{10}\Phi^i A^j C^k(W_\mu T_\mu \tau), \qquad W_\mu \in \Gamma_0(11^2) \\ &= 11^{-1}\sum_{\mu=0}^{10}\Phi^i A^j C^k(T_\mu\tau) = G(\tau), \end{aligned}$$

since all three functions are invariant under $\Gamma_0(11^2)$. This completes the proof of the first statement of Theorem 8.

That G can be expressed as a polynomial in A and C now follows in the same way as in the proof of Theorem 6. We set G equal to a polynomial

$$G = \Sigma a_{jk}A^jC^k, \qquad j = 0 \text{ or } 1, \quad k \geq j,$$

with undetermined a_{jk}. Then, equating coefficients of like powers of x in the expansions at the parabolic points, we obtain a set of linear equations analogous to (6.52). The first equation will involve only one coefficient, and each succeeding equation will add one new coefficient. Hence, the coefficients are determined uniquely. They are obviously rational numbers, since the expansions of all functions involved have rational coefficients. This establishes Theorem 8.

Making use of the remark in the lines following (7.8), we see that we have proved

THEOREM 9. *For every integer $\alpha \geq 1$, there is a "Ramanujan identity" of the form*

$$L(\tau; 11^\alpha) = P(A, C), \tag{8.6}$$

where L is defined in (7.12), (7.13), $A = A_{11}$ in (4.51), and C in (6.43). P is a polynomial in A and C with rational coefficients.

By using Theorem 7 we can put (8.6) in the form

$$(8.7) \qquad \sum_{l=0}^{\infty} p(q^\alpha l + \rho)x^{l+1} = P(A, C) \prod_1^\infty (1 - x^{\epsilon m})^{-1},$$

with the ρ, ϵ of (7.23), (7.22). Unfortunately, we have not been able to prove the *integrality* of the coefficients of P except when $\alpha = 1, 2$. Thus, for higher values of α, (8.7) gives no information concerning Ramanujan's congruence. The discussion for $\alpha = 2$ is given in the next section.

The following remark may be of interest. If $F(\tau)$ is an entire modular function of $\Gamma_0(q)$ and we subject U_qF to the transformation $\tau' = -1/q\tau$, we find, following the indications already given,

$$(8.81) \qquad qU_qF(-1/q\tau) - qU_qF(q\tau) = F(-1/q^2\tau) - F(\tau).$$

If, in addition, $F(\tau)$ has expansions in the *same* uniformizing variable (say $x = \exp 2\pi i\tau$) at both parabolic points, then U_qF will be entire modular on $\Gamma_0(q)$. Now suppose that $\Gamma_0(q)$ is a subgroup of genus zero. This is true for $q = 5, 7, 13$. Then there exists a univalent function

$$(8.82) \qquad \Phi_{q,r}(\tau) = \left\{ \frac{\eta(q\tau)}{\eta(\tau)} \right\}^r$$

where $r(q-1) \equiv 0 \ (24)$ and r is minimal. Moreover,

$$(8.83) \qquad \Phi_{q,r}(-1/q\tau) = q^{-r/2}\Phi_{q,r}^{-1}(\tau).$$

$F(\tau)$ and $U_qF(\tau)$ are polynomials in $\Phi_{q,r}(\tau)$, and this is true also of $F(\tau')$ and $U_qF(\tau')$, by (8.83). Substituting these expressions in (8.81) we obtain an algebraic equation connecting $\Phi_{q,r}(\tau)$ and $\Phi_{q,r}(q\tau)$. If this is irreducible, it is the *modular* equation of $\Phi_{q,r}(\tau)$. For the primes mentioned, these equations have been discussed by Rademacher [12].

Another approach to the modular equation is given in a forthcoming paper of the author [10].

9. The Ramanujan congruence for $\alpha = 2$. For $\alpha = 2$ we can determine the coefficients of P in (8.6) to an extent sufficient to establish the Ramanujan congruence. (6.6) can be written

$$L(\tau; 11) = 11^2(AC^2 - 11C^2) + 11(2(AC - 1) - 32C).$$

Also, by (7.13),

$$L(\tau; 11^2) = U_{11}L(\tau; 11).$$

The application of U_{11} to a power series simply replaces each coefficient a_n by a_{11n}. It is clear, then, that

$$L(\tau; 11^2) \equiv 2 \cdot 11U_{11}(AC - 1) + 11U_{11}C \pmod{11^2}.$$

But by Lemma 4, each coefficient of $AC - 1$ is a multiple of 11; hence,

$$L(\tau; 11^2) \equiv 11U_{11}C \pmod{11^2}.$$

By (7.2) Ramanujan's congruence for $q^a = 11^2$ is equivalent to

$$L(\tau; 11^2) \equiv 0 \pmod{11^2}.$$

Therefore, in order to prove Ramanujan's congruence for the modulus 11^2, *it is sufficient to show that*

$$U_{11}C \equiv 0 \pmod{11}. \tag{9.1}$$

This was done by a direct computation of the coefficients of the polynomial

$$U_{11}C(\tau) = Q(A(\tau), C(\tau)) = \sum_{j,k} b_{jk}A^j(\tau)C^k(\tau). \tag{9.2}$$

Since $U_{11}C$, A, and C have expansions (in $x = \exp 2\pi i\tau$) with integral coefficients, the statement (9.1) reduces to

$$b_{jk} \equiv 0 \pmod{11}. \tag{9.21}$$

It is therefore sufficient to determine the b_{jk} modulo 11.

The coefficients b_{jk} will be determined by considering the expansions at the parabolic point $\tau' = 0$, except for b_{00}, which is to be determined at $\tau = i\infty$. We therefore replace b_{00} by g, a constant which may vary with each occurrence, and determine g last.

The expansion of $U_{11}C$ at $\tau' = 0$ is, by (8.81) and (6.46),

$$11^3U_{11}C(\tau') = x^{-22} + 2x^{-11} + O(1), \qquad \tau' = -1/11\tau \tag{9.31}$$

and this we set equal to

$$g + 11^3\Sigma b_{jk}A^j(\tau')C^k(\tau') = g + 11^3\Sigma b_{jk}A^j(\tau)C^k(\tau'), \tag{9.32}$$
$$j = 0 \text{ or } 1, \quad k \geq j, \quad j + k \neq 0.$$

Putting $11^2C(\tau') = E(\tau)$ and $b_{jk} = c_{jk}11^{2k-3}$, we get

$$11^3U_{11}C(\tau') = g + \sum_{j,k} c_{jk}A^j(\tau)E^k(\tau). \tag{9.4}$$

We now see that (9.21) will be completely proved if we can show that

$$c_{j1} \equiv 0 \pmod{11^2}, \qquad j = 0, 1. \tag{9.41}$$

(The constant, g, will be determined separately.) This is true since $E(\tau)$ has an expansion in x with integral coefficients.

Clearly the term of highest degree in E in (9.4) will be E^{11}. We shall therefore need the power series of E and A to 22 terms. To compute E we have from (6.45)

$$2E = A^2 - 10A - 22 + B. \tag{9.5}$$

Since we must compute c_{jk} modulo 11^2, we shall need A and B modulo $2 \cdot 11^2$.

The calculation may be carried out as follows (see the table on p. 514). The coefficients of A are computed from its definition (4.51)

$$A = A_{11} = \frac{\theta_{11}^2(\tau)}{\psi_{11,2}(\tau)}, \qquad \psi_{11,2}(\tau) = \eta^2(\tau)\eta^2(11\tau).$$

First, $\eta^{-1}(\tau)\eta^{-1}(11\tau)$ is obtained from (3.4); then this is squared. Next, the coefficients of θ are calculated from (4.61) and footnote [16], and the resulting series squared. From these two series we obtain A and A^2 by multiplication and squaring.

The definition (4.51) of B is

$$B = B_{11} = \frac{G_{11,2}(\tau)}{\psi_{11,4}(\tau)}.$$

$\psi_{11,4}^{-1}$ is simply the square of $\psi_{11,2}^{-1}$. $G_{11,2}$ is defined by (4.31) and (4.23). By multiplying the last two series we get the coefficients of B.

The computation for each series (except that for θ^2 which has very small coefficients) was checked by recomputing the coefficients modulo $2 \cdot 11^3$.

Now, using (9.5), we obtain the coefficients of E, reduced modulo 11^2. We raise E to the eleventh power by the formula $E^{11} = E \cdot E \cdot E^2 \cdot E^4 E^3$, working modulo 11^2. In order to have a check on the coefficients of E^{11}, we observe that, by an elementary theorem on binomial coefficients, if

$$E(\tau) = x^{-2} \sum_{\nu=0}^{\infty} e_\nu x^\nu,$$

then

$$E^{11}(\tau) \equiv x^{-22} \sum_{\nu=0}^{\infty} e_\nu x^{11\nu} \pmod{11}.$$

Therefore,

$$\begin{aligned} E^{11}(\tau) \equiv E(11\tau) &= 11^2 C(-1/121\tau) \\ &= x^{-22} + 2x^{-11} + O(1) \pmod{11}, \end{aligned} \tag{9.6}$$

where we have taken the last expansion from (6.46) with τ replaced by 11τ. (9.6) shows that all the computed coefficients e'_ν of E^{11} must be multiples of 11 except $e'_{-22} \equiv 1 \pmod{11}$ and $e'_{-11} \equiv 2 \pmod{11}$. As a further check we computed E^{12} (modulo 11) in two ways:

$$E^{12} = E \cdot E^{11} = E^3 \cdot E^3 \cdot E^6.$$

Residues of the Coefficients of Certain Power Series

Power of x	Function								
	$x\psi_{11,2}^{-1}$	θ^2	x^2A	x^2A^2	$x^2\psi_{11,4}^{-1}$	$G_{11,2}$	x^2B	x^2E	$x^{22}E^{11}$
	modulo			$2\cdot 11^2$				mod 11^2	
0	1	1		1	1	1	1	1	1
1	2	4	1	12	4	−2	2	2	22
2	5	4	6	70	14	−18	−12	−12	−33
3	10	8	17	54	40	−56	−116	5	−55
4	20	20	46	105	105	96	−113	8	−55
5	36	16	116	72	10	−10	−120	1	0
6	65	32	10	−14	90	−20	−114	7	0
7	110	16	49	116	30	38	110	−11	22
8	−57	36	66	−56	−82	40	10	10	22
9	58	28	25	−4	98	−62	−12	−12	55
10	−3	40	−90	50	−56	−90	−18	−18	11
11	28	4	−39	72	94	−2	−22	−22	2
12	−41	64	6	−35	−107	26	−95	26	−11
13	86	40	−11	0	−82	−40	110	−11	−44
14	23	64	2	−60	90	100	−80	41	11
15	−118	56	75	68	−86	−38	44	44	−22
16	86	68	90	−83	−21	76	−15	−15	0
17	−112	40	30	−38	−114	94	−96	25	−22
18	108	100	76	−32	−6	−74	−66	55	0
19	−48	48	−76	20	−120	74	54	54	33
20	32	104	28	−117	23	−4	87	−34	55
21	−94	80	−36	−4	104	96	36	−46	−11

Setting $c_{jk} = 11f_{jk}$, we can now rewrite (9.4) as

$$(9.7)\qquad 11^2U_{11}C(\tau') \equiv 11^{-1}\cdot E^{11}(\tau) + \Sigma f_{jk}A^j(\tau)E^k(\tau) + g \pmod{11}.$$

We need to compute the f_{jk} only modulo 11, prove

$$(9.71)\qquad f_{j1} \equiv 0 \pmod{11},$$

and determine the constant g.

This is done exactly as in the proof of Theorem 6. We obtain a set of linear equations, analogous to (6.52), which can be solved stepwise for the f_{jk}.

The coefficients of the various powers of A and E are found quite readily since the computations are carried out modulo 11. The result is:

$$(9.8)\qquad 11^2U_{11}C(\tau') \equiv 11^{-1}\cdot E^{11} - (2E^{10}A + 8E^9A + 6E^9 + 4E^8A + 8E^8 + E^7A + 9E^7 + 7E^6A + 9E^6 + 2E^5A + 2E^5) + g \pmod{11}.$$

The computation was checked as follows:

start with:	$2EA + 8A + 6$
multiply by E:	$2E^2A + 8EA + 6E$
add $4A + 8$:	$2E^2A + 8EA + 6E + 4A + 8$
multiply by E:	$2E^3A + 8E^2A + 6E^2 + 4EA + 8E$,

etc. This procedure mixes the numbers quite thoroughly, and it is very unlikely that an error in the check would duplicate one in the original computation.

Recalling that $E(\tau) = 11^2C(\tau')$, we see that (9.8) implies (9.71). Now we have only to compute g. Let $\tau \to \tau'$ in (9.8). Since $A(\tau) = A(\tau')$, we get

$$\begin{aligned} U_{11}C(\tau) &\equiv g + 11Q(A, C) \pmod{11} \\ &\equiv g \pmod{11}, \end{aligned}$$

since $Q(A, C)$ is a polynomial with integer coefficients. Since

$$U_{11}C(\tau) = O(x)$$

we have

$$g \equiv 0(11),$$

which proves (9.1). By the remark in the lines preceding (9.1), this establishes

THEOREM 10. $p(121l + 16) \equiv 0 \pmod{121}$, $\qquad (l = 0, 1, 2, \cdots)$.

II. *Proofs of certain theorems.*

10. Proof of Theorem 1. That the functions

$$\psi_{q,r}(\tau) = \{\eta(q\tau)\eta(\tau)\}^r$$

are regular and zero-free in the interior of the upper half-τ-plane is clear from the definition (2.4) of $\eta(\tau)$. We consider their behavior under $V = (a\tau + b)/(c\tau + d) \in \Gamma_0(q)$. First, if $c = 0$, $V\tau = \tau + b$, and using (3.5) we obtain

$$\psi_{q,r}(V\tau) = \exp\{(\pi i/12)br(q + 1)\}\psi_{q,r}(\tau) = \psi_{q,r}(\tau),$$

because of condition (4.11). Thus (4.12) is proved for this case.

If $c > 0$, we apply (3.7) and find

$$\psi_{q,r}(V\tau) = N(a,b,c,d)\,(-i(c\tau+d))^r\psi_{q,r}(\tau)$$

with

$$(10.11)\qquad (-i)^r N(a,b,c,d) \equiv \exp(-\pi i r A)$$

$$(10.12)\qquad A = (s(a,c) - (a+d)/12c) + (s(a,c/q) - (a+d)/(12c/q)) + \tfrac{1}{2}.$$

Therefore, in order to establish (4.12), we must prove

$$(10.13)\qquad rA \equiv r\cdot\tfrac{1}{2}(1-(a/q)) \pmod 2.$$

a/q is the Legendre-Jacobi symbol.

We shall need the following properties of $s(h,k)$:
For $(h,k)=1$, we have

$$(10.21)\qquad 12hks(h,k) \equiv h^2+1 \pmod{\theta\cdot k}$$

$$(10.22)\qquad 12ks(h,k) \equiv 0 \pmod 3 \text{ if } 3\nmid k.$$

Here $\theta=1$ for $3\nmid k$ and $\theta=3$ for $3\mid k$.
For odd k we have

$$(10.23)\qquad 12ks(h,k) \equiv k+1-2(h/k) \pmod 8.$$

Let $k=2^\lambda l$, $\lambda\geq 0$, and let l and h be odd. Then

$$(10.24)\qquad 12hks(h,k) \equiv h^2+k^2+3k+1+2k(k/h) \pmod{2^{\lambda+3}}.$$

These properties appear as Theorems 17, 18, 19 in [13].

From (10.21) with $k=c$, c/q, we get

$$12ac\{s(a,c)-(a+d)/12c\} \equiv a^2+1-a(a+d) \equiv -bc \pmod{\theta c}$$

$$12ac\{s(a,c/q)-(a+d)/(12c/q)\} \equiv q(a^2+1)-qa(a+d) \equiv -qbc \pmod{\theta c}.$$

Hence, by addition,

$$12acrA \equiv -r(q+1)bc \equiv 0 \pmod{\theta c}.$$

Also, by (10.22)

$$12crA \equiv 0 \pmod 3 \text{ if } 3\nmid c.$$

Therefore, we have

$$(10.31)\qquad 12crA \equiv 0 \pmod{3c}.$$

Now suppose that c is odd. Then from (10.23) we obtain

$$12c\{s(a,c) - (a+d)/12c\} \equiv c + 1 - 2(a/c) - (a+d) \pmod 8,$$
$$12c\{s(a,c) - (a+d)/(12c/q)\}$$
$$\equiv c + q - 2q(a|c/q) - q(a+d) \pmod 8.$$

If we note that the Legendre-Jacobi symbols satisfy

$$(a/(c/q)) = (a/(c/q))(a/q^2) = (a/cq) = (a/c)(a/q),$$

we obtain

$$12crA \equiv 2cr + r(q+1) - 2r(a/c)(q(a/q)+1)$$
$$- r(q+1)(a+d) + 6rc \pmod 8$$
$$\equiv -2r(a/c)(q(a/q)+1) \pmod 8$$
$$\equiv -2r(a/c)(1-(a/q)) \pmod 8$$

by virtue of (4.11)

$$\equiv -2rc(a/c)(1-(a/q)) \pmod 8$$

since c is odd and $4|2(1-(a/q))$

(10.32) $$12crA \equiv -2rc(1-(a/q)) \equiv 6rc(1-(a/q)) \pmod 8.$$

By (10.31)

(10.33) $$12crA \equiv 6rc(1-(a/q)) \pmod{3c}.$$

From (10.32) and (10.33) we infer

$$12crA \equiv 6rc(1-(a/q)) \pmod{24c},$$

which is equivalent to (10.13).

Now let $c = 2^\lambda\gamma$, $\lambda > 0$, γ odd. Then, (10.24),

$$12ac\{s(a,c) - (a+d)/12c\}$$
$$\equiv a^2 + c^2 + 3c + 1 + 2c(c/a) - a(a+d) \pmod{2^{\lambda+3}}$$
$$12ac\{s(a,c/q) - (a+d)|(12c/q)\}$$
$$\equiv qa^2 + c^2/q + 3c + q + 2c(c/q|a) - qa(a+d) \pmod{2^{\lambda+3}},$$

from which it follows that

$$12acrA \equiv rc^2/q(q+1) + 6cr - rbc(q+1)$$
$$+ 2cr(c/a)((q/a)+1) + 6acr \pmod{2^{\lambda+3}}$$
$$\equiv \pm 2cr((q/a)+1) + 6cr(a+1) \pmod{2^{\lambda+3}}$$
$$\equiv \pm 2^{\lambda+1}r\gamma\{\pm(1+(q/a)) - (a+1)\}(2^{\lambda+3}),$$

(10.41) $$12acrA \equiv 2^{\lambda+1}r\gamma\{(q/a) - a\} \pmod{2^{\lambda+3}}.$$

This shows that

(10.42) $$12crA \equiv 0(2^{\lambda+3}) \text{ if } r \text{ is even},$$

since a must be odd if c is even, because of $(a, c) = 1$.

If r is odd, we must have $q \equiv -1 (4)$ and therefore

$$(q/a) = (a/q)(-1)^{(a-1)/2} \equiv a(a/q) \pmod 4.$$

Hence, from (10.41),

$$12acrA \equiv 2^{\lambda+1} r\gamma a((a/q) - 1) \pmod{2^{\lambda+3}},$$

so that

$$12crA \equiv \begin{cases} 0 \pmod{2^{\lambda+3}}, & \text{if } r \text{ is even or if } (a/q) = +1 \\ 2^{\lambda+2} \pmod{2^{\lambda+3}}, & \text{if } (a/q) = -1 \text{ and } r \text{ is odd.} \end{cases} \tag{10.43}$$

The congruence (10.43) proves (10.13) in case c is even. Thus the transformation equation (4.12) is established in all cases.

To prove (4.13) we use the formula (3.6).

$$\psi_{q,r}(-1/\tau) = \{\eta(-q/\tau)\eta(-1/\tau)\}^r = (-i\tau/q)^{r/2}(-i)^{r/2}\{\eta(\tau/q)\eta(\tau)\}^r$$
$$= (-iq^{-1/2}\tau)^r \psi_{q,r}(\tau/q).$$

(4.13)' now follows upon replacing τ by $q\tau$. This completes the proof of Theorem 1.

THEOREM 3. *Let q be a prime > 3, α an even integer, and*

$$\Phi(\tau; q^\alpha) = \frac{\eta(q^\alpha\tau)}{\eta(\tau)}, \qquad \Im\tau > 0.$$

Then

$$\Phi(V\tau) = \Phi(\tau)$$

for every $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *belonging to* $\Gamma_0(q^\alpha)$, i. e., *with* $c \equiv 0 \pmod{q^\alpha}$.

Proof. We shall make use of the transformation equations (3.5) and (3.7). If $c = 0$, $V\tau = \tau + b$, and

$$\Phi(V\tau) = \exp(\pi i/12)(q^\alpha - 1) \cdot \Phi(\tau) = \Phi(\tau), \tag{10.5}$$

since $24 \mid (q^\alpha - 1)$ by virtue of the conditions placed on q and α.

For $c > 0$, we apply (3.7) and find

$$\Phi(V\tau) = M(a, b, c, d)\Phi(\tau) \tag{10.61}$$

with

$$M = \exp(\pi i H) \tag{10.62}$$

and

$$H = (s(a, c) - (a + d)/12c) - (s(a, c/q^\alpha) - (a + d)/(12c/q^\alpha)). \tag{10.63}$$

We have therefore to show that

$$H \equiv 0 \pmod 2. \tag{10.64}$$

The proof of (10.64) is word for word the same as that of Lemma 5 in Rademacher [12] if we merely replace 2 by α.

Proof of Lemma 1. With

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad W_\mu = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},$$

we must have

$$\begin{pmatrix} a + \lambda c & b + \lambda d \\ qc & qd \end{pmatrix} = \begin{pmatrix} a_1 & a_1 + qb_1 \\ c_1 & c_1 + qd_1 \end{pmatrix},$$

or

$$\begin{aligned} a_1 &= a + \lambda c & a_1\mu + qb_1 &= b + \lambda d \\ c_1 &= qc & c_1\mu + qd_1 &= qd. \end{aligned} \tag{10.7}$$

Equations (10.7) determine c_1 and a_1 uniquely with $c \equiv 0 \pmod{q^2}$ and $(a_1, c_1) = 1$. From the equations on the right we find

$$\mu a \equiv b + \lambda d \pmod q. \tag{10.8}$$

Since $q \mid c$, $q \nmid a$, and (10.8) together with the condition $0 \leq \mu < q$ determines μ uniquely. Furthermore, $\mu_1 = \mu_2$ implies $(\lambda_1 - \lambda_2)d \equiv 0 \pmod q$, which, since $q \nmid d$, is equivalent to $\lambda_1 \equiv \lambda_2 \pmod q$, i. e., $\lambda_1 = \lambda_2$.

Now we find d_1 from (10.7):

$$d_1 = d - \mu c_1/q = d - \mu c.$$

If $c > 0$, we have

$$b_1 = \frac{a_1 d_1 - 1}{c_1} = \frac{(a + \lambda c)(d - \mu c) - 1}{qc} = \frac{c(b + \lambda d - \mu a) - \mu\lambda c^2}{qc}, \tag{10.9}$$

since $ad - bc = 1$. b_1 is an integer, as we verify from (10.8) and (10.9). If $c = 0$, we necessarily have $a = d = 1$, $a_1 = 1$, $d_1 = 1$, $c_1 = 0$, $\mu \equiv b + \lambda (q)$. From (10.7), $b_1 = (b + \lambda - \mu)/q$, an integer, and $a_1 d_1 - b_1 c_1 = 1$. Thus the lemma is proved.

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